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MATHEMATIK

Die explizite Bestimmung von einigen Klassen der assoziativen Ringe

von

FERENC A. SZÁSZ

Vorgelegt von K. BORSUK am 3 November 1958

Wir werden in der vorliegenden Note einige spezielle ringtheoretische Fragen und ihre Lösungen ohne Beweis erörtern.

I. Herr L. Fuchs hat alle Gruppen bestimmt [2], deren sämtliche Klassen von untereinander isomorphen Untergruppen endlich sind. Daher kann man auch eine ähnliche Frage für Ringe stellen, die aber noch einer Lösung harrt. Wir nennen einen Ring R mit der Eigenschaft E_1, wenn sämtliche endlich erzeugbare, von 0 und von R verschiedene Unterringe von R untereinander isomorph sind. (Für einen speziellen Fall vgl. [8]).

Wir haben nun den nachstehenden Satz beweist:

SATZ 1. *Alle Ringe mit Eigenschaft E_1 sind:*

1) *die Zeroringe mit torsionsfreier additiven Gruppe und mit dem Rang* 1;

2) *endliche Körper von der Ordnung p^q, wobei p und q Primzahlen sind;*

3) *direkte Summen $R = K_p + K_p$, wobei der endliche Primkörper K_p ein Ideal in R ist;*

4) *Ringe mit zyklischer additiver Gruppe und mit der Ordnung p oder p^2;*

5) *Zeroringe mit der Ordnung p oder p^2 (wobei p eine Primzahl ist);*

6) *Ringe $\{x\}$ mit dem Erzeugende x und mit den Relationen $px = x^3 = 0$.*

Bemerkung. Der in meiner Arbeit [8] gemeinte Ring $\{x\}$ (wobei $px = x(x^2+x)+(x^2+x) = 0$) ist kein Ring mit Eigenschaft E_1, da für $y = -2x-x^2$, $z = x+x^2$ notwendig $py = y^2-y = 0$ und $pZ = Z^2 = 0$ gelten.

II. Herr L. Rédei hat alle Ringe R mit einzigen Erzeugenden bestimmt [5], deren sämtliche Unterringe zweiseitige Ideale in R sind. Die Frage der Bestimmung aller Ringe R, derer sämtliche von R verschiedene Unterringe die Gestalt $R \cdot r (r \in R)$ haben, wurde in unserer Note [7] im Frühling 1956 aufgeworfen und im Sommer 1958 gelöst. Solche Ringe nennen wir kurz Ringe mit Eigenschaft E_2. Es gilt also der

Satz 2. *Alle Ringe mit Eigenschaft E_2 sind:*

1) *Restklassenringe $\mathfrak{Z}/(m)$ des Ringes $\mathfrak{Z}$ der ganzen rationalen Zahlen nach dem Ideal (m), wobei $m = 0, \pm 1, \pm 2, \ldots$;*

2) *Restklassenringe $(p)/(p^{k+1})$ des Ringes (p), welcher aus durch eine Primzahl p teilbaren ganzen rationalen Zahlen besteht, nach einem Ideal (p^{k+1}), wobei $k = 1, 2, \ldots$ ist;*

3) *Ringe $\{x\}$, mit den Relationen $px = x^3 = 0$;*

4) *Ringe $\{x, y\}$, mit den Relationen:*

$$px = py = x^2 = xy = y^2 - y = yx - x = 0 .$$

Bemerkungen. Der in der Arbeit [7] gemeinte Ring R, als eine direkte Summe $K_2 + K_2$ der zweielementigen Primkörper K_2, welche in R zweiseitige Ideale sind, besitzt die erwähnte Eigenschaft nicht, da aus $R = \{a\} + \{b\}$; $a + a = b + b = a^2 + a = b^2 + b = ab = ba = 0$ folgt, dass $\{a + b\}$ kein Linksideal in R ist.

Man kann auch zeigen, dass der Hauptsatz von [7] auch aus dem Satze 2. abgeleitet sein kann.

III. Etwas mehr kompliziert ist die explizite Lösung der folgenden Frage: es sind alle Ringe R zu bestimmen, welche eine solche Eigenschaft E_3 besitzen, dass sämtliche von R verschiedene Unterringe linksseitige Hauptideale $Rr + \mathfrak{Z}r (r \in R)$ von R sind.

Satz 3. *Alle Ringe mit Eigenschaft E_3 sind:*

A. *unter den Ringen mit torsionsfreier additiver Gruppe — nur die Ringe mit zyklischer Gruppe;*

B. *unter den Ringen mit gemischter additiver Gruppe — nur die Ringe $\{x\}$ mit den Relationen $n(x^2 - nlx) = (x - 1)(x^2 - nlx) = 0$, wobei $1 \in \mathfrak{Z}$, $n \in \mathfrak{Z}$, $l \in \mathfrak{Z}$ und $n > 1$ eine quadratfreie Zahl ist;*

C. *unter den Ringen mit unendlicher periodischer Gruppe — nur die Zeroringe des Typus p^∞;*

D. *unter den (endlichen und unendlichen) Ringen mit Abel'scher elementarer p-Gruppe *) sind nur:*

---

*) Eine Abel'sche p-Gruppe G wird eine elementare p-Gruppe genannt, wenn die Ordnung jedes von Null verschiedenen Elementes von G eine fixierte Primzahl p ist. Eine solche Gruppe G ist also die direkte Summe von zyklischen Gruppen der Ordnung p.

1) *sämtliche Unterringe* *) *der Ringe* $R = \{x\}$ *mit* $px = x^4 - x^3 = 0$,

2) *die Ringe* $R = \{x, y\}$, *mit* $px = py = x^2 = xy = y^2 = yx = 0$,

3) *die Ringe* $R = \{x, y\}$, *mit* $px = py = x^2 = xy = y^2 - y = yx - x = 0$,

4) *die Ringe* $R = \{x, y\}$, *mit* $px = py = x^3 = y^3 = xy = yx = x^2 - ky^2 = 0$, *wobei* $p \neq 2$ *eine Primzahl ist, ferner* $k \in \mathfrak{J}$ *und* $(-k)^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ *ist;*

5) *die Ringe* $R = \{x, y\}$, *wobei* $px = py = x^3 = y^3 = xy = yx - x^2 = x^2 - ky^2 = 0$, *und für* $p = 2$ *notwendig* $k \equiv 1 \pmod{p}$ *und für* $p \neq 2$ *notwendig die Kongruenz* $(k^2 - 4k)^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ *gilt,*

E. *unter den* p*-Ringen mit Abel'scher nicht elementarer* p *Gruppe — nur*

1) *die Ringe* $R = \{x\}$, *mit* $p^n x = x^2 - p^f x = 0$, $n \geqslant 2$ *und* $0 \leqslant f \leqslant n$,

2) *die Ringe* $R = \{x\}$, *wobei* $p^n x = p(x^2 - p^f x) = (x-1)(x^2 - p^f x) = 0$, $1 \in \mathfrak{J}$, $n \geqslant 2$, $1 \leqslant f \leqslant n$; *ist,*

F. *unter den endlichen Ringen mit additiver nicht* p*-Gruppe — nur die Ringe* $R = \sum R_p$, *wobei das Ideal* R_p *aus sämtlichen Elementen von* p*-Potenzordnung von* R *besteht, und* R_p *zu einem unter* E. *oder unter* D. 1) *gemeinten Ringe isomorph ist.*

Bemerkung. Aus dem Satze 3. kann man mehr als 16 nicht-triviale Folgerungen ableiten. Der Beweis benützt den Wedderburn-Artin'schen Struktursatz, ferner unseren früheren Satz [6], welche eine Verallgemeinerung der Arbeit [9] von T. Szele ist. In meiner Arbeit ([6], S. 331) muss man anstatt $RN = 0$, $R = N$, $R^2 = 0$ richtig als $\mathfrak{J}N = 0$, $R = N$, $\mathfrak{J}^2 = \mathfrak{J}R = 0$ lesen. Später folgt in [6] die weggebliebene englische Aussage: "Namely if $R/\mathfrak{J}$ had been a skewfield with the unity $e + \mathfrak{J}$, we should have obtained $0 \neq e^2 = e^3 \in Re^2 = R$, and $0 \neq \mathfrak{J} = K \cdot e^2 = Ke^3 = \mathfrak{J}e \leqslant \mathfrak{J}R = 0$ for some additive subgroup K of R^+, which is a contradiction".

IV. L. Fuchs und T. Szele haben beweist [3], das ein Ring R dann und nur dann ein halbeinfacher Ring mit Minimalbedingung für Linksideale ist, wenn jedes Linksideale von R ein rechtsseitiges Einzelement enthält. Wir werden diesen Satz hier verallgemeinern, indem wir aus der Tatsache, dass in einem Ringe R mit Minimumbedingung für Linksideale sämtliche Linksideale in der Gestalt $L = Rc + L_1$ darstellbar sind **), wobei $e^2 = e$ (nicht notwendig $= 0$) und L_1 ein Nillinksideal ist, die folgende Definition ermitteln.

Definition. *R wird ein Ring mit Eigenschaft E_4 genannt, wenn jedes Linksideal von R die Gestalt $L = Re + L_1$ hat* **), *wobei L_1 mit lauter nilpotenten Elementen ist, und $e^2 = e$.*

Als Resultat haben wir bekommen:

Satz 4. *Ein Ring R hat die Eigenschaft E_4 dann und nur dann, wenn R ein zweiseitiges Nilideal N besitzt, und R/N ein halbeinfacher Ring mit Minimalbedingung für Linksideale ist.*

*) Diese sind auch explizit darstellbar.

**) Die Summe ist direkt.

Als eine Folgerung, ergibt sich daraus der Hauptsatz von [3]. Der in dem Satze 4. erwähnte Ring ist ein halb-primärer Ring. Der unendliche zyklische Zeroring besitzt die Eigenschaft E_4 ohne die gemeinte Minimalbedingung für Linksideale.

MATHEMATISCHES INSTITUT, EÖTVÖS LORÁND UNIVERSITÄT, BUDAPEST

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MATHEMATICS

Metrtic Semilattices and Their Connections with the Measure Theory

by

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Presented by K. BORSUK on January 21, 1959

It is a well known fact that there are many reasons for which partial order is an important feature in the measure theory as well as in the theory of integral [5]. In fact, the whole theory, as presented by Daniell, Stone, McShane and others, may be based upon partial order. This paper is an attempt to go still further and to bring to light the algebraic and topological structures of the special partial ordering involved in the two theorie, i. e. that of measure and that of integral. We consider some abstract algebras called pseudometric semilattices, which, to some extent, are similar to topological groups, and we list several simple notions and statements which can be interpreted in a natural way in both these theories. For the sake of illustration, as immediate consequences, we obtain the extension theorems of Caratheodory [3], [4] and of Daniell [2]. This point of view gives the advantage of unique treatment of the notions of measure and integral from the very beginning, and gives rise to a compact, abstract theory of semilattices.

A set L, with associative and commutative operation $ab \in L$ for $a, b \in L$, is said to be a *semilattice*, if the operation is idempotent, i. e. if for each $a \in L$ we have $aa = a$.

A net (a_n) of elements of L [6] is said to be *monotone*, if

$$a_n a_m = a_m \text{ for each } n \leqslant m$$

or

$$a_n a_m = a_n \text{ for each } n \leqslant m .$$

The monotone nets of the first kind will be called α-monotone nets, and of the second kind β-monotone nets.

A monotone net $(a_n) \subset L$ is said to be *bounded*, if there is $a_0 \in L$ (named a *bound* of (a_n)) such that $a_n a_0 = a_0$, if (a_n) is an α-monotone net or, $a_n a_0 = a_n$, if (a_n) is a β-monotone one.

An element a_0 is said to be a strict bound of (a_n), $a_0 = \operatorname{sb}_n a_n$, if a_0 is a bound of (a_n) and for every other bound a' we have $a'a_0 = a'$, in the case (a_n) is α-monotone, and $a'a_0 = a_0$ in the case (a_n) is β-monotone.

In each semilattice L a partial ordering relation can be introduced as follows

$$a \leqslant b \text{ if and only if } ab = b\,.$$

It is easy to see that $\leqslant$ is transitive and semi-antisymmetric ($a \leqslant b$ and $b \leqslant a$ imply $a = b$).

The element $ab \in L$ coincides with the least upper bound of $a, b \in L$, with respect to the relation $\leqslant$.

This remark leads to an alternative definition of the semilattice. Let L be a set with a transitive and semi-antisymmetric relation. We call $\leqslant$ a partial ordering relation. The set L is a semilattice if and only if each pair $a, b \in L$ has the least upper bound: l. u. b. (a, b) with respect to the partial ordering relation $\leqslant$.

We obtain in L the semilattice operation setting

$$xy \overset{\text{df}}{=} \text{l. u. b. } (x, y)\,.$$

This shows that the definition of monotone sequences, as well as the definition of a bound and that of strict bound of a monotone sequence, coincide with those considered in the theory of partially ordered sets.

A semilattice L with a pseudometric ϱ [6] is said to be *a pseudometric semilattice*, if

$$\varrho(ac, bc) \leqslant \varrho(a, b) \quad \text{for each} \quad a, b, c \in L\,. \tag{1}$$

Condition (1) implies trivially the continuity of the operation in $L(\varrho)$. In accordance with the following remark we shall very often be able to consider a metric semilattice instead of a pseudometric one.

Remark 1. *To each pseudometric semilattice L there corresponds a metric semilattice $\tilde{L}(\tilde{\varrho})$ and an isomorphism h of L onto $\tilde{L}$ such that*

$$\tilde{\varrho}(hx, hy) = \varrho(x, y) \quad \text{for each} \quad x, y \in L\,.$$

In fact, setting $a \equiv b$ for $a, b \in L$ if and only if $\varrho(a, b) = 0$ we obtain the family of rest classes $\tilde{L}$.

Introducing $\tilde{a}\tilde{b} \overset{\text{df}}{=} \widetilde{ab}$, $\tilde{\varrho}(\tilde{a}, \tilde{b}) \overset{\text{df}}{=} \varrho(a, b)$ and $ha = \tilde{a}$, we arrive at the desired result.

The following statement is an simple consequence of condition (1).

Statement 1. *If $L(\varrho)$ is a metric space and if, in some dense subset $K \subset L$, a semilattice operation is introduced in such a way that $K(\varrho)$ be a metric semilattice, then the operation can be uniquely extended over L in such a way that $L(\varrho)$ be a metric semilattice.*

Thus, by virtue of Statement 1 concerning the problem of completion of a metric semilattice, we have the following theorem:

THEOREM 1. *Each metric semilattice $L(\varrho)$ has a unique extension to a complete metric semilattice $\bar{L}(\varrho)$ such that L is dense in $\bar{L}$.*

Condition (1) implies

STATEMENT 2. *In a metric semilattice, the limit of a given monotone sequence is the strict bound of that sequence.*

For pseudometric semilattices Statement 2 is generally false. There exists a pseudometric semilattice $L(\varrho)$ with a monotone sequence that has both a limit a_1 and the strict bound a_2, and $\varrho(a_1, a_2) > 0$.

We say that a pseudometric semilattice $L(\varrho)$ is *proper*, if, for each monotone Cauchy sequence $(a_n) \subset L$, we have

$$\lim_n \varrho(a_n, \operatorname{sb}_m a_m) = 0\,, \tag{2}$$

provided $\operatorname{sb}_m a_m$ exists in L.

STATEMENT 3. *A pseudometric semilattice is proper if and only if, for each Cauchy monotone sequence $(a_n) \subset L$, we have*

$$\varrho(\operatorname{sb}_m a_m, a_1) \leqslant \sum_{n=1}^{\infty} \varrho(a_n, a_{n+1})\,, \tag{3}$$

provided $\operatorname{sb}_m a_m$ exists in L.

Statement 2 implies easily

Remark 2. *If each monotone Cauchy sequence in metric semilattice $L(\varrho)$ has a limit, then $L(\varrho)$ is proper.*

There are examples to the effect, that, given the assumption of this remark, the condition of convergence of each monotone Cauchy sequence cannot be dropped. The following formula is of basic significance to the theory of pseudometric semilattices

$$\varrho(a_k a_{k+1} \ldots a_n, a_l a_{l+1} \ldots a_m) \leqslant \sum_{i=p}^{\infty} \varrho(a_i, a_{i+1})\,, \tag{4}$$

where $k, l \leqslant p \leqslant n, m$.

Let $L(\varrho)$ be a proper pseudometric semilattice and let each monotone Cauchy sequence of elements from L have the strict bound in L. If $\sum_{n=1}^{\infty} \varrho(x_n, x_{n+1}) < +\infty$, then by virtue of (4), $b_{n,m} = a_{n+1} a_{n+2} \ldots a_{n+m}$ $(m = 1, 2, \ldots)$ is an a-monotone Cauchy sequence for arbitrary but fixed n, and

$$\lim_k \varrho(\operatorname{sb}_m b_{n,m}, b_{n,k}) = 0\,.$$

Further, $c_n = \operatorname{sb}_m b_{n,m}$ is a β-monotone Cauchy sequence with

$$\lim_m \varrho(c_m, a_m) = \lim_k \varrho(\operatorname{sb}_m c_m, c_k) = 0$$

and hence (a_n) tends to $\operatorname{sb}_m c_m$.

Thus, calling a pseudometric semilattice $L(\varrho)$ *properly-complete*, if each monotone Cauchy sequence has the strict bound in L which is a limit of this sequence, we obtain the following three theorems.

THEOREM 2. *Each properly-complete pseudometric semilattice is complete.*

THEOREM 3. *A metric semilattice is properly-complete if and only if it is complete.*

THEOREM 4. *If L_0 is a dense subsemilattice of a properly-complete pseudometric semilattice L, and if K consists of the strict bounds (taken in L) of all α-monotone sequences of elements from L_0, then L coincides with the set of the strict bounds (taken in L) of all β-monotone sequences of elements from K.*

An extension of a given pseudometric semilattice $L(\varrho)$ to a complete pseudometric semilattice $\bar{L}(\varrho)$, such that L is dense in $\bar{L}$, is not unique.

The pseudometric ϱ of a pseudometric semilattice $L(\varrho)$ is said to be *additive* on a subset K of L if

$$\varrho(abc, c) = \varrho(abc, bc) + \varrho(bc, c) \text{ for each } a, b, c \in K. \tag{5}$$

If K is a subsemilattice of L, then, using the partial ordering notation, (5) can be written as follows:

$$\varrho(x, z) = \varrho(x, y) + \varrho(y, z) \text{ for } x \leqslant y \leqslant z \text{ from } K. \tag{5'}$$

If ϱ is additive on the whole of L, then we say that $L(\varrho)$ is an *additive-pseudometric* semilattice.

We have the following theorem:

THEOREM 5. *If ϱ is additive on some subsemilattice K of a pseudometric semilattice $L(\varrho)$, then ϱ is also additive on the closure of K.*

STATEMENT 4. *Let (a_n) be a monotone net of elements from a pseudometric semilattice L. If ϱ is additive on (a_n), and if $\varrho(a_n, a_m)$ is bounded, then (a_n) satisfies the Cauchy condition.*

From Statement 4 there immediately follows

THEOREM 6. *If (a_n) is a monotone sequence in a proper pseudometric semilattice L, and if ϱ is additive on (a_n) then $\lim_m \varrho(a_m, \operatorname{sb}_n a_n) = 0$, provided $\operatorname{sb}_n a_n$ exists in L.*

Theorem 6 yields the following theorem, being in a sense the converse of Theorem 4.

THEOREM 7. *Let L_0 be a closed subset of a proper pseudometric semilattice L and let ϱ be additive on L_0.*

If (x_n) is a monotone sequence of elements of L_0 and if x_0 is the strict bound of (x_n) taken in L, then $x_0 \in L_0$.

Statement 4 yields an interesting theorem that is closely connected with Theorem 1 of [7].

THEOREM 8. *Each monotone net (a_n) of elements from a complete additive-metric semilattice L, in which all the countable monotone subnets are bounded, has a strict bound in L, and there is a countable subnet of (a_n) that has the same strict bound as (a_n).*

The measure-theoretical concepts yield two important examples of pseudometric semilattices.

A. Let $\mathfrak{M}$ be a ring of subsets of a given set T and let μ be a positive valued (including the value $+\infty$) function on $\mathfrak{M}$, satisfying the conditions:

$$A \subset B \cup C \text{ implies } \mu(A) \leqslant \mu(B) + \mu(C) .$$

We set

$$\mathfrak{M}_\mu \overset{\text{df}}{=} \{A \in \mathfrak{M} : \mu(A) < +\infty\} \quad \text{and} \quad \varrho_\mu(A, B) \overset{\text{df}}{=} \mu(A \cup B - A \cap B)$$

It is easy to verify that $\mathfrak{M}_\mu(\varrho_\mu)$ is a pseudometric semilattice with respect to the operation of union. The same is true if the intersection is taken instead. If μ is σ-subadditive, i. e. if

$$A \subset \bigcup_n A_n \text{ implies } \mu(A) \leqslant \sum_n \mu(A_n) ,$$

then $\mathfrak{M}_\mu(\varrho_\mu)$ is proper.

The pseudometric ϱ_μ is additive on some subring $\mathfrak{N} \subset \mathfrak{M}_\mu$ if and only if $\mu(A \cup B) = \mu(A) + \mu(B)$ for each $A, B \in \mathfrak{N}$ with $A \cap B = 0$.

We call μ *a measure* if μ is additive on $\mathfrak{M}$ and if, for each $(A_n) \subset \mathfrak{M}$ with $A_{n+1} \subset A_n$, the condition: $\bigcap_n A_n = 0$ implies $\inf_n \mu(A_n) = 0$, is satisfied.

THEOREM A. (Measure Extension Theorem). *Each measure μ defined on a ring $\mathfrak{M}$ can be uniquely extended to a measure $\bar{\mu}$ on a least σ-ring $\bar{\mathfrak{M}}$, containing $\mathfrak{M}$. The space $\mathfrak{M}_\mu$ is dense in the space $\mathfrak{M}_\mu(\varrho_{\bar{\mu}})$, where $\overline{\mathfrak{M}}_\mu(\varrho_{\bar{\mu}})$ and M_μ are defined as above. Moreover, the space $\mathfrak{M}_{\bar{\mu}}$ is properly-complete.*

In order to prove Theorem A., it will be sufficient to define $\bar{\mu}$ over $\bar{\mathfrak{M}}$ in the usual way:

$$\bar{\mu}(A) \overset{\text{df}}{=} \inf \Big\{ \sum_{n=1}^{\infty} \mu(A_n) : A \subset \bigcup_{n=1}^{\infty} A_n, (A_n) \subset \mathfrak{M} \Big\} \quad A \in \bar{\mathfrak{M}} .$$

Clearly, $\bar{\mu}$ is σ-subadditive and thus the Theorem follows easily from Theorems 5, 6 and 7.

It is very interesting that, from Theorems 5, 6 and 7, the following important theorem can easily be obtained:

THEOREM A*. *If T is a metric space and μ is a metric outer measure* (see [1] p. 48) *on the set $\overline{\mathfrak{M}}$ of all subsets of T, then $\bar{\mu}$ is additive over σ-ring of all the Borel sets $\mathfrak{B} \subset \mathfrak{M}$.*

B. This example concerns a problem which is analogous to the previous one, but this time for the notion of integral. A set $\mathfrak{F}$ of non-negative valued function (including the value $+\infty$) on a given set T is said to be a *special lattice* of non-negative functions on T if, for each $f, g \in \mathfrak{F}$ the pointwise sum $f+g$, maximum $f \vee g$, minimum $f \wedge g$ and for $g \leqslant f$, also the difference $f-g$ all belong to $\mathfrak{F}$ (we set $\infty - \infty \overset{\text{df}}{=} 0$). If each monotone sequence of functions from $\mathfrak{F}$ has the pointwise strict bound in $\mathfrak{F}$, then we say that $\mathfrak{F}$ is a special σ-lattice.

Let $\mathcal{J}$ be a positive valued (including the value $+\infty$) function on $\mathfrak{F}$ satisfying the condition

$$f \leqslant g+h \text{ implies } \mathcal{J}f \leqslant \mathcal{J}g + \mathcal{J}h .$$

We set

$$\mathfrak{F}_{\mathcal{J}} \overset{\text{df}}{=} \{f : \mathcal{J}f < +\infty\}$$

and

$$\varrho_{\mathcal{J}}(f, g) \overset{\text{df}}{=} \mathcal{J}(|f-g|) .$$

It is easy to verify that $\mathfrak{F}(\varrho_{\mathcal{J}})$ is a pseudometric semilattice with respect to the operation of pointwise maximum or with respect to pointwise minimum. If $\mathcal{J}$ is *σ-subadditive*, i. e. if

$$f \leqslant \sum_n^{\infty} f_n \quad \text{implies} \quad \mathcal{J}f \leqslant \sum_n^{\infty} \mathcal{J}f_n$$

then $\mathfrak{F}_{\mathcal{J}}(\varrho_{\mathcal{J}})$ is proper.

The pseudometric $\varrho_{\mathcal{J}}$ is additive on some special sublattice $\mathfrak{E} \subset \mathfrak{E}_{\mathcal{J}}$ if and only if $\mathcal{J}(f+g) = \mathcal{J}f + \mathcal{J}g$ for each $f, g \in \mathfrak{E}$.

We call $\mathcal{J}$ an integral if $\mathcal{J}$ is additive on $\mathfrak{F}_{\mathcal{J}}$ and if, for each $(f_n) \subset \mathfrak{F}$ with $f_{n+1} \leqslant f_n$, we have

$$\inf_n f_n = 0 \text{ implies } \inf_n \mathcal{J}f_n = 0 .$$

THEOREM B. (Extension Theorem for Integral). *Each integral $\mathcal{J}$ defined on a special lattice $\mathfrak{F}$ of non-negative functions on a space T can be uniquely extended to an integral $\bar{\mathcal{J}}$ over the least special σ-lattice $\bar{\mathfrak{F}}$, containing $\mathfrak{F}$.*

The space $\mathfrak{F}_{\mathcal{J}}$ is dense in $\bar{\mathfrak{F}}_{\mathcal{J}}(\varrho_{\mathcal{J}})$, where $\mathfrak{F}_{\mathcal{J}}$ and $\overline{\mathfrak{F}}_{\bar{\mathcal{J}}}(\varrho_{\bar{\mathcal{J}}})$ are defined as above. Moreover, the space $\bar{\mathfrak{F}}_{\mathcal{J}}(\varrho_{\bar{\mathcal{J}}})$ is properly-complete.

Similarly as in case A. it will be sufficient to define $\mathcal{J}$ over $\bar{\mathfrak{F}}$ setting

$$\bar{\mathcal{J}}f = \inf\Big\{\sum_n \mathcal{J}f_n : f \leqslant \sum_n f_n,\ (f_n) \subset \mathfrak{F}\Big\} f \in \bar{\mathfrak{F}}.$$

Clearly, $\bar{\mathcal{J}}$ is σ-subadditive, and, as in the previous case, Theorem B follows easily from Theorems 5, 6 and 7.

The examples A and B have a common generalization. An abstract object named a special lattice can be defined which admits the operation of extension, similarly as in the case A. and B. The generalization of both Theorems A. and B. will be given in a separate paper.

This is a preliminary report of a paper dedicated to Professor K. Kuratowski on the 40th anniversary of his scientific activity. The work will be published in full in Fundamenta Mathematicae.

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MATHEMATICS

On Regular Embeddings of a Group with Differential Operators

by

W. SŁOWIKOWSKI

Presented by K. BORSUK on January 21, 1959

It is more or less obvious which conditions should be satisfied by a mapping $y = Dx$ of an arbitrary algebraic ring into itself, so that it can be reasonably thought of as an operation of differentiation. Such conditions are, for instance, $Dx + y = Dx + Dy$ and $Dxy = xDy + yDx$. The generalization of the notion of a differential operator on arbitrary Abelian groups presented in this paper brings to light properties that are quite different in character. In this stage of generalization the only properties still left are: 1) additivity, 2) some simple conditions on domains which are precisely stated below. With these assumptions we are lead step-by-step to a general theory of those deeply hidden properties of differential operators that are of fundamental importance to Schwartz distributions and other theories of generalized functions.

It should be noted that this paper is something of a generalization of the algebraic subject-matter of all the papers quoted.

A group with differential operators $(\mathcal{R}, X)$, briefly a *d-group* $(\mathcal{R}, X)$, is a group X with a family of subgroups G_A, $A \in \mathcal{R}$, chosen in X so that, to each $A \in \mathcal{R}$ and $x \in G_A$, there corresponds a unique element Ax called the composition of A and x, subject to the condition that the mapping $y = Ax$ is a homomorphism of G_A into X, for each $A \in \mathcal{R}$.

Since, at least in this paper, the set $\mathcal{R}$ will be fixed once and for all we shall drop the letter $\mathcal{R}$ and write simply X instead of $(\mathcal{R}, X)$.

Let X and X' be two d-groups. A homomorphism h of X into X' is said to be *compatible* if, for each $A \in \mathcal{R}$, we have $hG_A \subset G'_A$, and for each $x \in G_A$,

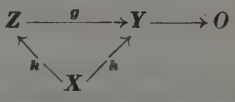
$$h(Ax) = A(hx).$$

This is expressed by writing h instead of h, and we say that h maps X into X'. Let $X_0 = hX$ and let $X_{n+1} = \{\sum_{i=1} A_i h x_i : A_i \in \mathcal{R}, x_i \in G_A \cap X_n\}$. If the group X' coincides with $\bigcup_n X'_n$, we say that the

homomorphism h maps X almost onto X'. In this case X' is called the almost-image of X. A d-group X is said to be closed if, for each $A \in \mathcal{R}$, we have $G_A = X$.

THEOREM 1. *Let h be a homomorphism of a d-group X into a d-group Y. In order that h should map X almost onto Y it is necessary and sufficient that, for each d-group Z that is closed and for each homomorphism k of X into Z and each homomorphism g of Z into Y that g maps Z onto Y, if $h = gk$.*

Making use of the diagrams this can be formulated as follows: *h maps X almost onto Y if and only if each commutative diagram with Z closed can be completed to the diagram with exact row*:



In what follows we shall assume once and for all that $\mathcal{R}$ is a ring with the two-sided identity I.

A d-group X is said to be *a modul* if X is closed and if for each $A, B \in \mathcal{R}$ and $x \in X$ we have:

$$A(Bx) = (AB)x \qquad \text{and} \qquad (A+B)x = Ax + Bx\,.$$

Speaking of a d-group X, we must keep in mind that the set of operators $\mathcal{R}$ is an arbitrary ring, but fixed once for all and that, as a rule, the ring operations in $\mathcal{R}$ are neither correlated with the operation of addition in X, nor with the operation of superposition of operators from $\mathcal{R}$, $A(Bx)$. This situation radically changes when X is assumed to be a modul. Given a modul X, the ring operations are closely connected with the elementwise addition, $Ax+Bx$, and the elementwise superposition $A(Bx)$ of the operators from $\mathcal{R}$. This paper is a study of the moduls that are almost-images of a given d-group.

Let X be a d-group. A pair composed of a modul Z and a homomorphism h of X into Z is said to be a regular embedding of the d-group X. Let, as usual, $\mathcal{R} \otimes X$ denote the tensor product of the additive group of $\mathcal{R}$ and of the group X. If we introduce a composition on generators of $\mathcal{R} \otimes X$ setting

$$A(B \otimes x) \overset{\text{df}}{=} (AB) \otimes x\,,$$

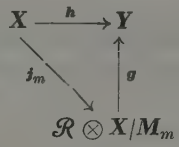
then the natural extension of this composition over the whole group $\mathcal{R} \otimes X$ leads to a modul $\mathcal{R} \otimes X$.

The group X can be isomorphically embedded into $\mathcal{R} \otimes X$. The natural mapping $x \to I \otimes x$ is an isomorphism. In general, this isomorphism is not compatible. The set of all differences $I \otimes (Ax) - A \otimes x$, where $A \in R$, $x \in G_A$, generates a submodul M_m of $\mathcal{R} \otimes X$.

The submodul $M_m \subset \mathcal{R} \otimes X$ is named the *minimal regularizer* of X. Let us denote by j_m the superposition of the embedding of X into $\mathcal{R} \otimes X$ and the natural mapping of $\mathcal{R} \otimes X$ onto $\mathcal{R} \otimes X/M_m$. Clearly, j_m is compatible. The following theorem makes it clear, why this particular name "minimal regularizer" was preferably chosen.

THEOREM 2. *If X is a d-group, Y is a modul and h is a homomorphism of X into Y, then there exists a homomorphism g of $\mathcal{R} \otimes X/M_m$ into Y such that $h = gj_m$.*

The situation can be sketched on the commutative diagram:



A submodul $M \subset \mathcal{R} \otimes X$ is said to be *compatible*, provided M contains the minimal regularizer M_m. The compatible submodul M is said to be a *regularizer* of X.

Let M be a submodul of $\mathcal{R} \otimes X$ and let us denote by j_M the homomorphism of X almost onto $X_M \overset{\text{df}}{=} \mathcal{R} \otimes X/M$. This homomorphism is a superposition of the mapping which imbeds X into $\mathcal{R} \otimes X$ and the natural mapping of $\mathcal{R} \otimes X$ onto X_M.

THEOREM 3. *The mapping j_M is compatible if and only if the submodul M is compatible.*

Thus, each regularizer M of X induces a regular embedding $j_M : X \to X_M$ of X.

Let X be a d-group, Y a modul, and h a homomorphism of X into Y. Let in the following

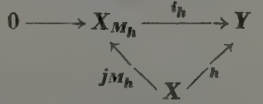
$$M_h = \Big\{ \sum_{i=1}^{n} A_i \otimes x_i : \sum_{i=1}^{n} A_i h x_i = 0 \Big\}.$$

THEOREM 4. *The mapping h is compatible if and only if M_h is compatible.*

The regularizer M_h of a d-group X which corresponds to a given compatible homomorphism h is said to be the quasi-kernel of h.

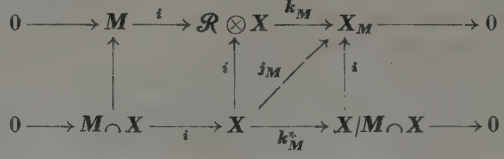
We obtain an isomorphism i_h of X_{M_h} into Y, setting on generators of $\mathcal{R} \otimes X/M_h$: $i_h(A \otimes x) \overset{\text{df}}{=} Ahx$.

THEOREM 5. *Let $h : X \to Y$ be a regular embedding of X. The following diagram is commutative and the row is exact:*



The mapping h is an isomorphism if and only if $M_h \cap X = 0$.

To each regularizer M of X there corresponds the following commutative diagram with exact rows:



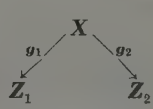
The mappings k_M, k_M^*, j_M are natural mappings. Given an arbitrary operator from $\mathcal{R}$, its domain in $X/M \cap X$ is simply the image of the corresponding domain in X.

All the mappings considered, except the embedding of X into $\mathcal{R} \otimes X$, are compatible. The quasi-kernel of j_M coincides with M and, thus, j_M is an isomorphism if and only if $M \cap X = 0$. Therefore, a regularizer M of X is named *an extensor*, if $M \cap X = 0$.

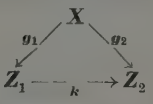
Let X be a d-group and let $g_1 : X \to Z_1$ and $g_2 : X \to Z_2$ be two regular embeddings of X such that g_1 maps X almost onto Z_1.

THEOREM 6. *In order that $M_{g_1} \subset M_{g_2}$ it is necessary and sufficient that there exist a compatible homomorphism k of Z_1 into Z_2 such that $g_2 = kg_1$. The homomorphism k is uniquely determined by g_1 and g_2. The homomorphism k maps "onto" if and only if g_2 maps "almost onto".*

Making use of diagrams, this can be formulated as follows: *The diagram*



can be completed to a commutative diagram

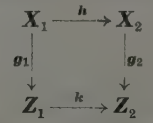


if and only if $M_{g_1} \subset M_{g_2}$.

Let h be a homomorphism of a d-group X into a d-group X', and let M' be a regularizer of X'. We set

$$(h)^{-1}M' \overset{\text{df}}{=} \left\{ \sum_{i=1}^{n} A_i \otimes x_i : \sum_{i=1}^{n} A_i \otimes hx_i \in M' \right\}.$$

Consider Theorem 6. If we replace X by $X_1 \overset{h}{\cdot} X_2$, where h maps X_1 into X_2 we obtain that a necessary and sufficient condition for the diagram



being completed with a compatible k, is that $M_{g_1} \subset (h)^{-1}M_{g_2} = M_{g_2 h}$.

If this condition is satisfied, then there is exactly one k that completes the diagram to a commutative one.

Let X be a d-group and let $\mathfrak{I}$ be a function defined on $\mathcal{R}$ with values $\mathfrak{I}(A)$ in the set of all subgroups of X. In accordance with [2], [3], [5] and [6], we shall call $\mathfrak{I}$ an ideal of X if and only if

$$A \otimes x \in M_{\mathfrak{I}} \quad \text{implies} \quad x \in \mathfrak{I}(A),$$

where $M_{\mathfrak{I}}$ is the least compatible submodul of $\mathcal{R} \otimes X$ containing the set $\{A \otimes x : A \in \mathcal{R}, x \in \mathfrak{I}(A)\}$.

It is clear that, for fixed $A \in \mathcal{R}$, the subgroup $\mathfrak{I}(A)$ is something like the set of elements that are mapped onto zero in $\mathcal{R} \otimes X / M_{\mathfrak{I}} = X_{M_{\mathfrak{I}}}$. The regularizer $M_{\mathfrak{I}}$ that corresponds to an ideal $\mathfrak{I}$ is said to be *induced* by $\mathfrak{I}$.

Let N be a regularizer of X and let

$$\mathfrak{I}_N(A) \overset{\text{df}}{=} \{x \in X : A \otimes x \in M, A \in \mathcal{R}\}.$$

THEOREM 7. *The function $\mathfrak{I}_N$ is an ideal of X and $N \supset M_{\mathfrak{I}_N}$.*

We shall call $\mathfrak{I}_N$ the ideal *induced* by N. There are examples in which the equality $N = M_{\mathfrak{I}_N}$ does not hold, and thus, for some X there are different compatible moduls M, M' such that $\mathfrak{I}_M = \mathfrak{I}_{M'}$.

In several important problems of extension, especially in those concerning distributions, we are dealing with the ring $\mathcal{R} = Z(\mathcal{S})$ generated by a semigroup $\mathcal{S}$.

The ring $Z(\mathcal{S})$ is a free commutative group generated by $\mathcal{S}$ with multiplication defined on the set $\mathcal{S}$ of generators according to the usual multiplication in $\mathcal{S}$ and for each $A \in Z(\mathcal{S})$ we have

$$G_A = \bigcap_{i=1}^{n} G_{A_i}, \qquad Ax = \sum_{i=1}^{n} n_i A_i x,$$

where $A_i \in \mathcal{S}$, $A_i \neq A_j$ for $i \neq j$ and $A = \sum_{i=1}^{n} n_i A_i$, $n_i \neq 0$, $x \in G_A$.

In this case ideals are uniquely determined by their values on the set of generators $\mathcal{S}$.

In papers [2], [3], [5], and [6] there are many examples of such ideals which are uniquely determined by fixing their values merely on $\mathcal{S}$. In papers [2], [3] and [6] it is assumed that all the operators from $\mathcal{S}$ map their domains onto the whole group, which permits to obtain a simple characterization of the ideal and, in this case, for each regularizer N, the equality $N = M_{\mathfrak{I}_N}$ holds. But in papers [4] and [5] this assumption is dropped and in the case of [5] a very complicated characterization of ideals is given. The case of periodic distributions was discussed in [6].

Periodic distributions contain a d-group X that is a regularizer N of X such that $N \neq M_{\mathfrak{Z}_N}$.

I wish to thank Professor MacLane for his valuable advice during the preparation of this paper.

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MATHEMATICS

Categories and Representations of Groups with Differential Operators

by

W. SŁOWIKOWSKI

Presented by K. BORSUK January 21, 1959

The notion of a regular embedding of a group with differential operators has been introduced in [3].

A group with differential operators X, briefly a d-group, is an algebraic group X together with a family $\mathcal{R}$ of homomorphisms A each of which maps its domain G_A into the group X. The domains G_A are always assumed to be subgroups of X. It often happens that the family $\mathcal{R}$ turns out to be an algebraic ring, though the ring operation in $\mathcal{R}$ may have nothing to do with the elementwise operations in $\mathcal{R}$.

A regular embedding of a d-group X into a modul Y over the same ring $\mathcal{R}$ is, roughly speaking, a mapping h of the group X into the group Y, which, in addition to being a group homomorphism, is also compatible with the elementwise operations in the family $\mathcal{R}$ of all d-operators, i. e.

$$hAx = Ahx\,,$$

provided the left side exists.

Throughout this paper it is taken for granted that the reader is familiar with [3].

It also happens that, while in search of regular embedding of a given d-group in a particular class of moduls, we admit only special homomorphisms of this d-group into moduls of this particular class, as in [4].

For instance, if a d-group X has the property of X being a topological group, then it is very natural to consider only such regular embeddings $h: X \to Y$ of X, where Y is also a topological group, each operator from $\mathcal{R}$ is continuous over Y and h is a continuous mapping of X into Y.

The purpose of this note is to develop an algebraic theory of regular embeddings of a given d-group that are subject to some special conditions.

Let X be a fixed d-group over a ring $\mathcal{R}$ with identity and let Ξ be a fixed property of regular embeddings of X. In place of "x has the property P" let us rather write "$x \in P$" and this will be read as x belongs to P. Of course, in general, the set of objects with the property P does not exist in the classical sense.

Let $Z \in \Lambda$ denote such a property of a modul Z that, for some homomorphism h, we have $h: X \to Z \in \Xi$.

We assume that, to each Z_1, Z_2, there corresponds a set $\mathcal{H}(Z_1, Z_2)$ of homomorphisms of Z_1 into Z_2 in such a way that, if $h_1 \in \mathcal{H}(Z_1, Z_2)$ and $h_2 \in \mathcal{H}(Z_2, Z_3)$, then $h_2 h_1 \in \mathcal{H}(Z_1, Z_3)$, and that for each $Z \in \Lambda$ the identical mapping of Z onto Z belongs to $\mathcal{H}(Z, Z)$.

Further, we assume that

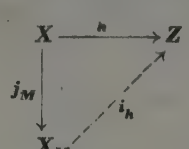
$$\text{if} \quad h: X \to Z \in \Xi \quad \text{and} \quad g \in \mathcal{H}(Z, Y), \quad \text{then also} \quad gh: X \to Y \in \Xi.$$

Clearly, the above introduced notions lead to a special example of category [1].

If X is a d-group and X is a topological group, then it is very natural to assume that $h: X \to Z \in \Xi$ means that Z is a topological group, that all the operators from $\mathcal{R}$ are continuous over Z and that h is a continuous mapping of X into Z.

Since, according to the definition of regular embedding, $h: X \to Z$, the modul Z corresponds uniquely to h, we shall often omit $X \to Z$ and write briefly h instead of $h: X \to Z$.

Let M be a regularizer of a d-group X (see [3]). By virtue of the Theorem 5 of [3] there exists a unique isomorphism i_h that corresponds to a given regular embedding h of X as presented in the commutative diagram



The isomorphism i_h is said to be *a specialization* of X_M corresponding to the regular embedding h. Clearly, this is a one-to-one correspondence between regular embeddings and specializations.

With each regularizer M of X we associate a property Ξ_M. $h \in \Xi_M$ means that $h \in \Xi$ and that the quasi-kernel of h is equal to M. This can be written formally as follows:

$$\Xi_M \overset{\text{df}}{=} \{h \in \Xi : M_h = M\}.$$

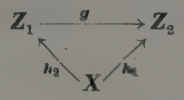
To each $h \in \Xi_M$ there corresponds a specialization i_h of X_M. From the algebraic point of view it is sufficient to consider only the regular embeddings j_M induced by regularizers. In connection with the previous

topological example, the specialization i_h of X_M, where $h \in \Xi_M$, induces a topology in X_M, which so far was topologically indifferent.

We introduce a partial ordering relation among Ξ regularizations of X setting for $h_1, h_2 \in \Xi$:

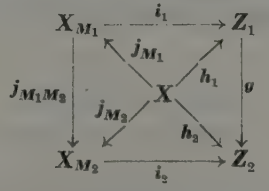
$h_1 \leqslant h_2$ if and only if there is $g \in \mathcal{H}(Z_1, Z_2)$ such that $h_2 = gh_1$.

This can be represented on the commutative diagram:



Clearly, this relation is, in general, not semi-antisymmetric, i. e. from $h \leqslant g$ and $g \leqslant h$ it does not follow that $h = g$.

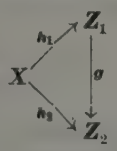
Let $h_1 \in \Xi_{M_1}$ and $h_2 \in \Xi_{M_2}$. If $h_1 \leqslant h_2$, then, by Theorem 6 of [3], we have $M_1 \subset M_2$ and again it follows from Theorem 6 that the diagram



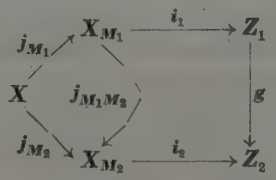
is commutative.

Here, the isomorphisms i_1 and i_2 are specializations corresponding to h_1 and h_2, respectively.

Hence, the commutative diagram



can be replaced by the following one:



where the dotted arrows represent the algebraic component of the inequality $h_1 \leqslant h_2$.

Thus, we have the following

THEOREM 1. *The inequality* $h_1 \leqslant h_2$ *holds if and only if* $M_1 \subset M_2$, *and the diagram*

$$\begin{array}{ccc} X_{M_1} & \xrightarrow{i_1} & Z_1 \\ {\scriptstyle j_{M_1M_2}}\downarrow & & \\ X_{M_2} & \xrightarrow{i_2} & Z_2 \end{array}$$

where i_1 *and* i_2 *are specializations of* X_{M_1} *and* X_{M_2} *corresponding to* h_1 *and* h_2 *respectively, can be completed to a commutative diagram*

$$\begin{array}{ccc} X_{M_1} & \xrightarrow{i_1} & Z_1 \\ {\scriptstyle j_{M_1M_2}}\downarrow & & \downarrow{\scriptstyle g} \\ X_{M_1} & \xrightarrow{i_2} & Z_2 \end{array}$$

where $g \in \mathcal{H}(Z_1, Z_2)$.

A regular embedding $h_0 \in \Xi$ that is minimal in Ξ, i. e. such that for each $h \in \Xi$ we have $h_0 h$, is named a Ξ minimal regular embedding of X.

Clearly, each Ξ minimal regular embedding of X belongs to Ξ_{M_0}, where $M_0 \overset{\text{df}}{=} \bigcap_{h \in \Xi} M_h$.

From Theorem 1 we obtain the following

Remark 1. If the specification i_0 corresponding to some $h_0 \in \Xi$ satisfies the condition:

to each specification i corresponding to $h \in \Xi$ there correspond some $g \in \mathcal{H}(Z_0, Z)$ such that the diagram

$$\begin{array}{ccc} X_M & \xrightarrow{i_0} & Z \\ {\scriptstyle j_{M_0M}}\uparrow & & \uparrow{\scriptstyle g} \\ X_{M_0} & \xrightarrow[i_0]{} & Z_0 \end{array}$$

is commutative,

then h_0 is a Ξ minimal embedding of X.

In this section we introduce the notion of representations of a d-group into a fixed Abelian group.

Let X be a d-group and let U be a given Abelian group. A mapping φ of $\mathcal{R} \times X$ into U is said to be a representation of X into U if the following conditions are satisfied:

1. for each fixed $x \in X$ $H_x(A) = \varphi(A, x)$ is a homomorphism of the additive group of $\mathcal{R}$ into U and, for each fixed $A \in \mathcal{R}$ $h_A x = \varphi(A, x)$ is a homomorphism of the group X into U.

2. for each $A, B \in \mathcal{R}$ and $x \in G_B$ we have

$$\varphi(A, Bx) = \varphi(AB, x).$$

For instance, if $h: X \to Z$ is a regular embedding of X, then the function

$$\varphi_h(A, x) \overset{\text{df}}{=} Ahx,$$

where $A \in \mathcal{R}$ and $x \in X$, is a representation of X into $U = Z$.

Each representation of a d-group X into a fixed Abelian group U induces a regularizer

$$M_\varphi = \Big\{ \sum_{i=1}^{n} AB_i \otimes x_i : A, B_i \in \mathcal{R}, x_i \in X \quad \text{for} \quad i = 1, 2, \ldots, n; \sum_{i=1}^{n} \varphi(B_i, x_i) = 0 \Big\} \quad \text{of } X.$$

Let Φ be a family of representation of a d-group X into a fixed Abelian group U such that Φ contains $\psi(A, x) = \varphi_1(BA, x) + \varphi_2(A, x)$, for each

$$\varphi_1, \varphi_2 \in \Phi \quad \text{and} \quad B \in \mathcal{R}.$$

The family Φ is a modul over $\mathcal{R}$ with respect to the addition

$$(\varphi_1 + \varphi_2)(A, x) \overset{\text{df}}{=} \varphi_1(A, x) + \varphi_2(A, x)$$

and the composition

$$B\varphi(A, x) \overset{\text{df}}{=} \varphi(BA, x).$$

The modul Φ is called an U-adjoint of X.

To each U-adjoint modul Φ there corresponds a regularizer

$$M_\Phi \overset{\text{df}}{=} \bigcap_{\varphi \in \Phi} M_\varphi.$$

THEOREM 2. *In order that regularizer M_Φ be an extensor it is necessary and sufficient that*

$$\text{if } \varphi(I, x) = 0 \text{ for all } \varphi \in \Phi \text{ then } x = 0.$$

We note that if Φ_1 and Φ_2 are two U-adjoint moduls of X and if $\Phi_2 \subset \Phi_1$, then $M_2 \subset M_1$.

Let Φ be a U-adjoint modul and let X_Φ be the group of compatible homomorphism of Φ into U with pointwise addition, generated by the set of all homomorphisms $f_{A,x}(\varphi) = \varphi(A, x)$ where A, x runs over $\mathcal{R} \times X$.

Setting $Af_{B,x} = f_{AB,x}$ we have introduced the operation of composition in X_Φ and thus X_Φ has turned out to be a modul over $\mathcal{R}$.

The following theorem is very much like an abstract existence theorem for distributions:

Theorem 3. *The mapping*

$$h_\Phi x \overset{\text{df}}{=} f_{I,x} \quad \text{for} \quad x \in X,$$

where I is the identity of the ring $\mathcal{R}$, is a compatible homomorphism of X almost onto X_Φ. Moreover, the quasi-kernel of h_Φ is equal to M_Φ.

The method of construction of the regular embedding $X_\Phi: X \to X_\Phi$ is due to Schwartz and hence such regular embeddings will be called embeddings of the Schwartz type.

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MATHÉMATIQUE

Certaines inégalités relatives aux solutions de l'équation parabolique linéaire normale

par

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Présenté par T. WAŻEWSKI le 31 décembre, 1958

1. Nous établissons ici certaines inégalités relatives aux solutions de l'équation

$$(1)\quad \mathcal{F}[u] = \sum_{i,j=1}^{m} a_{ij}(X,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{m} b_j(X,t)\frac{\partial u}{\partial x_j} + c(X,t)\,u - \frac{\partial u}{\partial t} = f(X,t),$$

déterminées dans un domaine non borné de l'espace-temps $\mathcal{E}^{m+1}$ des variables $x_1, x_2, \ldots, x_m, t$. On désigne par $\mathcal{E}^m$ l'espace des variables $x_1, x_2, \ldots, x_m$, par X — le point aux coordonnées $x_1, x_2, \ldots, x_m$. Nous supposons que les coefficients de l'équation (1) sont définis dans un domaine D situé dans une couche $X \in \mathcal{E}^m$, $0 < t < T$ $(T > 0)$de $\mathcal{E}^{m+1}$. La frontière $F(D)$ de ce domaine se compose, par hypothèse, des domaines non-bornés S_0 et S_T à m dimensions des hyperplans $t = 0$ et $t = T$ et d'une surface σ qui n'est tangente à aucune caractéristique $t = \text{const}$.

Les inégalités que nous allons établir permettent d'obtenir ensuite des évaluations des solutions de l'équation (1), données dans le travail [4] sous des hypothèses beaucoup moins restrictives.

2. Désignons par $\sum$ l'ensemble $S_0 + \sigma$. En particulier, σ peut être un ensemble vide, S_0 et S_T seront alors des hyperplans $t = 0$ et $t = T$, on aura donc $\sum = S_0$.

Soit Γ_R la surface cylindrique $x_1^2 + \ldots + x_m^2 = R^2$ et D_R l'ensemble ouvert séparé de D par la surface Γ_R. Nous supposons que — le nombre R_0 étant assez grand — pour $R > R_0$, l'ensemble D_R se compose d'un nombre fini de domaines bornés dont les frontières sont composées des domaines des hyperplans $t = 0$, $t = T$ et des surfaces latérales non tangentes aux caractéristiques $t = \text{const}$.

3. On suppose que les coefficients $a_{ij}(X,t)$ de l'équation (1) sont bornés dans le domaine D et que la forme

$$\sum_{i,j=1}^{m} a_{ij}(X,t)\lambda_i\lambda_j$$

est définie positive pour $(X,t) \in D$. Quant aux coefficients $b_j(X,t)$ et $c(X,t)$, nous supposons l'existence des nombres non négatifs $\alpha, \beta, \bar{\alpha}$ et $\bar{\beta}$ tels que l'on ait

$$(2) \qquad \begin{aligned} c(X,t) &\leqslant \alpha \sum_{j=1}^{m} x_j^2 + \beta \qquad (X,t) \in D, \\ |b_j(X,t)| &\leqslant \bar{\alpha} \sum_{j=1}^{m} |x_j| + \bar{\beta} \end{aligned} \quad \text{pour} \quad (j = 1, 2, \ldots, m).$$

Nous allons démontrer le théorème suivant:

THÉORÈME 1. *Soit $u(X,t)$ une solution de l'équation* (1), *de classe C^2 dans le domaine D, continue et de classe E_2* (*voir* [2]) *dans l'ensemble $D+\sum$. Si l'on a $f(X,t) \leqslant 0$ (ou $f(X,t) \geqslant 0$) dans le domaine D et si la fonction $u(X,t)$ est non-négative* (*resp. non-positive*) *aux points de l'ensemble $\sum$, elle est non-négative* (*resp. non-positive*) *pour $(X,t) \in D$. En particulier, si D est une couche: $X \in \mathcal{E}^m$, $0 < t < T$, et si $f(X,t) \leqslant 0 (f(X,t) \geqslant 0)$ dans D, l'inégalité $u(X,0) \geqslant 0$ (resp. $u(X,0) \leqslant 0$) pour $X \in \mathcal{E}^m$ entraîne l'inégalité $u(X,t) \geqslant 0$ (resp. $u(X,t) \leqslant 0$) dans D.*

Si l'on a $f(X,t) \equiv 0$ dans D et $u(X,t) = 0$ sur $\sum$, on a $u(X,t) \equiv 0$ dans D.

Démonstration. La fonction $u(X,t)$ étant de classe E_2 dans l'ensemble $D+\sum$, on a

$$(3) \qquad |u(X,t)| \leqslant M \exp k_0 \sum_{j=1}^{m} x_j^2 \quad \text{pour} \quad (X,t) \in D+\sum,$$

où M et k_0 sont deux nombres positifs.

Dans la démonstration du théorème nous allons nous servir d'une fonction $H(X,t,k)$ du point (X,t) de $\mathcal{E}^{m+1}$ et du paramètre $k > 0$, définie pour $(X,t) \in \bar{D}_k$, où $\bar{D}_k$ est la fermeture d'un domaine D_k séparé de D par l'hyperplan $t = h(k)$ et situé dans la couche $-\infty < x_i < +\infty$ $(i = 1, \ldots, m)$, $0 < t < h(k)$, $h(k)$ étant une fonction de k qui sera définie dans la suite. La fonction $H(X,t;k)$ va jouir des propriétés suivantes:

$$(4) \qquad H(X,t;k) \geqslant \exp k \sum_{j=1}^{m} x_j^2 > 0 \quad \text{pour} \quad (X,t) \in \bar{D}_k,$$

$$(5) \qquad \mathcal{L}[H] < 0 \quad \text{pour} \quad (X,t) \in D_k.$$

Une telle fonction a été introduite dans mon travail antérieur [2]. Soit $A = \max_{i,j \leqslant m} [\sup_D |a_{ij}(X,t)|]$ et soit $\mathfrak{A}$ le plus petit nombre positif tel que l'on ait, pour chaque vecteur $\vec{\lambda}(\lambda_1, \lambda_2, \ldots, \lambda_m)$, l'inégalité

$$\sum_{i,j=1}^{m} a_{ij}(X,t)\lambda_i\lambda_j \leqslant \mathfrak{A} \sum_{j=1}^{m} \lambda_j^2 \quad \text{pour} \quad (X,t) \in D .$$

La fonction $H(X,t;k)$ est déterminée par la formule

$$H(X,t;k) = \exp\left[\frac{k}{1-\mu(k)t} \sum_{j=1}^{m} x_j^2 + \nu(k)t\right] \tag{6}$$

avec

$$\mu(k) = 4k\mathfrak{A} + 2m\bar{a} + \alpha + \lambda^2, \qquad \nu(k) = N + \frac{km\bar{\beta}^2}{\lambda^2} + \beta + \frac{2kmA}{\mu\delta}, \tag{7}$$

λ et N étant des nombres positifs choisis d'une manière arbitraire, δ — un nombre positif, inférieur à $1/\mu(k)$ et arbitrairement petit. Choisissons

$$h(k) = \frac{1}{\mu(k)} - \delta .$$

On a alors aux points du domaine D_k l'inégalité

$$1 - \mu(k)t > \mu(k)\delta . \tag{8}$$

On déduit de (2), (7) et (8) l'inégalité

$$\frac{1}{H}\mathcal{F}[H] \leqslant -N < 0 \text{ *)} .$$

Nous reprenons ensuite un raisonnement exposé par M. Picone dans le travail [5]. Choisissons une valeur du paramètre $k > k_0$ et posons

$$u(X,t) = v(X,t) \cdot H(X,t;k) . \tag{9}$$

La fonction $v(X,t)$ satisfait dans le domaine D_k à l'équation linéaire parabolique

$$\sum_{i,j=1}^{m} a_{ij}(X,t)\frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{j=1}^{m} \bar{b}_j(X,t)\frac{\partial v}{\partial x_j} + \bar{c}(X,t)v - \frac{\partial v}{\partial t} = f(X,t) , \tag{10}$$

où, en vertu de (5), on a:

$$\bar{c}(X,t) = \frac{1}{H}\mathcal{F}[H] < 0 , \qquad \bar{f}(X,t) = \frac{1}{H}f(X,t) \geqslant 0 . \tag{11}$$

*) Les détails des calculs sont analogues à ceux du par. 3 de [2].

Soit $P_0(X_0, t_0)$ un point quelconque de D_k. Considérons une suite d'ensembles $D_k^{(n)}$ séparés de D_k par les surfaces cylindriques Γ_n ($n = 1, 2, \ldots$) définies par les équations $x_1^2 + \ldots + x_m^2 = n^2$. Choisissons le nombre n_0 de sorte que, pour $n > n_0$, les ensembles $D_k^{(n)}$ contiennent le point P_0 et que la frontière de l'ensemble $D_k^{(n)}$ se compose d'une partie $S_0^{(n)}$ située sur la caractéristique $t = 0$, d'une partie située sur la caractéristique $t = h(k)$ et d'une "surface latérale" $\sigma_k^{(n)}$ de $D_k^{(n)}$ composée d'un sous-ensemble de Γ et d'un sous-ensemble de σ. On a, par hypothèse, $u(X, t) \geqslant 0$ et par suite $v(X, t) \geqslant 0$ sur $S_0^{(n)}$. D'autre part, il résulte de (3), (8) et (4) (pour $k > k_0$) que, ε étant un nombre positif quelconque, on peut choisir le nombre $n_1 \geqslant n_0$ de sorte que l'inégalité $n > n_1$ entraîne l'inégalité $|v(X, t)| < \varepsilon$ pour $(X, t) \in \sigma_k^{(n)}$. Soit $\sum_k^{(n)} = S_0^{(n)} + \sigma_k^{(n)}$. On a

$$v(X, t) > -\varepsilon \quad \text{pour} \quad (X, t) \in \sum_k^{(n)}, \; n > n_1 . \tag{12}$$

Or, il résulte des inégalités (11) et des propriétés connues des extrema des solutions des équations paraboliques [1], p. 372, [5]) que l'inégalité (12) subsiste dans l'ensemble $D_k^{(n)}$. On a en particulier $v(X_0, t_0) > -\varepsilon$. Le nombre ε étant arbitraire, on a $v(X_0, t_0) \geqslant 0$ et par suite $u(X_0, t_0) \geqslant 0$. Or, P_0 étant un point arbitraire de D_k, on a $u(X, t) \geqslant 0$ dans D_k.

Si le nombre $h(K)$ est inférieur à T (voir par. 1), on divise le domaine D en domaines partiels par les plans $t = p \cdot h(k) (p = 1, 2, \ldots, r)$ et on établit de proche en proche l'inégalité $u(X, t) \geqslant 0$ dans ces domaines.

4. On peut démontrer maintenant le théorème III du travail [4] sans recourir à l'hypothèse (A) (voir [4]). On a notamment un théorème suivant:

THÉORÈME 2. *Supposons que les coefficients de l'équation* (1) *satisfassent aux hypothèses du par.* 3 *et que l'on ait* $c(X, t) \leqslant 0$ *et* $f(X, t) \leqslant 0$ (*ou* $f(X, t) \geqslant 0$). *Soit* $u(X, t)$ *une solution de cette équation de classe* C^2 *dans le domaine* D, *continue et de classe* E_2 *dans l'ensemble* $D + \sum$. *Si l'on a* $u(X, t) \geqslant -M$ (*resp.* $u(X, t) \leqslant M$) *sur* $\sum$, *où* M *est un nombre non négatif, on a aussi* $u(X, t) \geqslant -M$ (*resp.* $u(X, t) \leqslant M$) *dans le domaine* D.

Pour la démonstration on pose (pour le cas où $f(X, t) \leqslant 0$)

$$\overline{u}(X, t) = u(X, t) + M .$$

Il est évident que l'on a

$$\mathcal{L}[\overline{u}] = f(X, t) + M c(X, t) \leqslant 0 \quad \text{dans} \quad D$$

et $\overline{u}(X, t) \geqslant 0$ sur $\sum$. On a donc, d'après le Théorème 1, $\overline{u}(X, t) \geqslant 0$ et par suite $u(X, t) \geqslant -M$ dans D.

On peut ensuite établir les inégalités des corollaires 1 et 2 de la "Remarque" du par. 3 de [4] sans avoir récours à l'hypothèse (A).

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MATHEMATICS

Periodic Solutions of a System of Two Ordinary Differential Equations

by

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Z. Opial [1] has proved the following theorem:

THEOREM Z. *Let the equation*

$$x'' = f(t, x, x') \tag{1}$$

where the function $f(t, x, x')$ is continuous for all t, x, x', satisfy the uniqueness condition.

Suppose that

$$f(t+1, x, y) \equiv f(t, x, y) \tag{2}$$

$$f(t, \bar{x}, y) \leqslant f(t, \bar{\bar{x}}, y) \quad \text{whenever} \quad \bar{x} < \bar{\bar{x}} \tag{3}$$

and assume that there exists a solution $x_0(t)$ of (1) such that

$$|x_0(t)| \leqslant M, |x_0'(t)| \leqslant M, (M < +\infty), \quad \text{for} \quad t \geqslant 0. \tag{4}$$

Under the assumptions given above the Eq. (1) has at least one periodic solution of period 1.

The purpose of this paper is to present a simple proof of a slight generalization of Theorem Z, by a method different from that of Z. Opial.

THEOREM C. *Suppose we are given a system of two differential equations*

$$x' = g(t, x, y), \quad y' = h(t, x, y), \tag{5}$$

where the functions $g(t, x, y)$ and $h(t, x, y)$ are continuous in the whole space.

Let the following conditions be satisfied:

a) *system (5) satisfies the uniqueness condition,*

b) *the equalities*

$$g(t+1, x, y) = g(t, x, y), \quad f(t+1, x, y) = f(t, x, y)$$

hold for arbitrary t, x, y,

c) $g(t, x, \bar{y}) \leqslant g(t, x, \bar{\bar{y}})$ *if* $\bar{y} < \bar{\bar{y}}$ *and* $h(t, \bar{x}, y) \leqslant h(t, \bar{\bar{x}}, y)$ *if* $\bar{x} < \bar{\bar{x}}$.

We assume furthermore that there exists a solution $(x_0(t), y_0(t))$ *of* (5) *which is bounded in* $[0, +\infty)$.

Our assumptions imply that (5) *possesses a periodic solution* $(x_*(t), y_*(t))$ *of period* 1, *such that*

$$x_0(t) = x_*(t) + o(1), \quad y_0(t) = y_*(t) + o(1). \tag{6}$$

The proof of Theorem C is based on two lemmas.

LEMMA 1. *Let the assumptions* a) *and* b) *of Theorem* C *be satisfied, and let* $(x_0(t), y_0(t))$ *be a bounded solution (for* $t \geqslant 0$) *of system* (5). *Suppose that there exists a number* t^* *such that each of the functions*

$$x_0(t) - x_0(t+1), \quad y_0(t) - y_0(t+1)$$

is always non-positive for $t \geqslant t^*$ *or non-negative for* $t \geqslant t^*$.

Then system (5) *has a periodic solution* $(x_*(t), y_*(t))$ *of period* 1, *and* $(x_*(t), y_*(t))$ *satisfies* (6).

Proof. We shall restrict ourselves to the case when

$$x_0(t) - x_0(t+1) \geqslant 0, \quad y_0(t) - y_0(t+1) \leqslant 0 \quad \text{for} \quad t \geqslant t^*. \tag{7}$$

The reasoning in other cases is similar. Now, let

$$x_n(t) \overset{\text{df}}{=} x_0(t+n), \quad y_n(t) \overset{\text{df}}{=} y_0(t+n) \quad (n = 1, 2, \ldots). \tag{8}$$

It is easy to verify that $(x_n(t), y_n(t))$ $(n = 1, 2, \ldots)$ is a solution of system (5). Since

$$x_{n+1}(t) - x_n(t) = x_0(n+1+t) - x_0(n+t)$$

and

$$y_{n+1}(t) - y_n(t) = y_0(n+1+t) - y_0(n+t),$$

we get from (7) that

$$x_{n+1}(t) - x_n(t) \geqslant 0 \quad \text{and} \quad y_{n+1}(t) - y_n(t) \leqslant 0 \quad \text{for} \quad t \geqslant 0 \quad \text{and} \quad n \geqslant t^*.$$

Hence, the sequences $\{x_n(t)\}$ and $\{y_n(t)\}$ are monotonic for $n \geqslant t^*$. But the functions $x_n(t), y_n(t)$ are uniformly bounded and equicontinuous for $t \geqslant 0$. Therefore, the sequences $\{x_n(t)\}, \{y_n(t)\}$ are convergent for $t \geqslant 0$ and convergence is uniform on every finite interval. Hence,

$$x_n(t) \rightrightarrows x_*(t); \quad y_n(t) \rightrightarrows y_*(t) \quad \text{for} \quad 0 \leqslant t \leqslant 1. \tag{9}$$

It is easy to see that $(x_*(t), y_*(t))$ is a solution of (5). Since

$$x_*(1) = \lim_{n \to +\infty} x_n(1) = \lim_{n \to +\infty} x_{n+1}(0) = x_*(0)$$

and

$$y_*(1) = \lim_{n \to +\infty} y_n(1) = \lim_{n \to +\infty} y_{n+1}(0) = y_*(0),$$

the solution $(x_*(t), y_*(t))$ is periodic of period 1. Owing to (9), for $\varepsilon > 0$ exists N such that

$$|x_*(t)-x_n(t)|+|y_*(t)-y_n(t)| < \varepsilon \quad \text{for} \quad n \geqslant N, 0 \leqslant t \leqslant 1, \tag{10}$$

but it follows from (8) and from the periodicity of $x_*(t)$ and $y_*(t)$ that

$$|x_*(t)-x_n(t)|+|y_*(t)-y_n(t)| = |x_*(t+n)-x_0(t+n)|+|y_*(t+n)-y_n(t+n)|.$$

We therefore get by (10)

$$|x_*(t)-x_0(t)|+|y_*(t)-y_0(t)| < \varepsilon \quad \text{for} \quad t \geqslant N.$$

This inequality proves relations (6) to be true and at the same time completes the proof of Lemma 1.

LEMMA 2. *Suppose that* (5) *satisfies the conditions* a) *and* c) *of Theorem* C. *Let* $(x_1(t), y_1(t))$ *and* $(x_2(t), y_2(t))$ *be two arbitrary solutions of* (5) *determined for* $t_0 \leqslant t < T$, $(T \leqslant +\infty)$.

Then there exists a number t^*, $t_0 \leqslant t^* < T$, *such that each of the differences*

$$x_1(t)-x_2(t), \quad y_1(t)-y_2(t)$$

is always non-positive for $t^* \leqslant t < T$ *or non-negative for* $t^* \leqslant t < T$.

Proof. Let us put

$$\varphi(t) \overset{\text{df}}{=} (x_1(t)-x_2(t))(y_1(t)-y_2(t)).$$

We shall prove that the following implication holds:

(I) if $\varphi(\bar{t}) = 0$, then $\varphi(t) \geqslant 0$ for $\bar{t} \leqslant t < T$.

First, the validity of (I) will be proved under the following additional assumptions:

$$g(t, x, \bar{y})-g(t, x, \bar{\bar{y}}) < 0 \quad \text{if} \quad \bar{y} < \bar{\bar{y}} \tag{11}$$

and

$$h(t, \bar{x}, y)-h(t, \bar{\bar{x}}, y) < 0 \quad \text{if} \quad \bar{x} < \bar{\bar{x}}. \tag{12}$$

In order to do so, let us calculate the first derivative of $\varphi(t)$. We have

$$\varphi'(t) = \big(g(t, x_1(t), y_1(t))-g(t, x_2(t), y_2(t))(y_1(t)-y_2(t))\big) + \big(h(t, x_1(t), y_1(t))-h(t, x_2(t), y_2(t))\big)(x_1(t)-x_2(t)).$$

From the previous formula we conclude by (11) and (12) that the following implication is true

(II) if $\varphi(\bar{t}) = 0$, then $\varphi'(\bar{t}) > 0$.

It is easy to see that (I) is a consequence of (II). This fact implies the truth of (I) if (11) and (12) are satisfied.

Suppose now that $g(t, x, y)$ and $h(t, x, y)$ satisfy only c). Then the right hand members of a system

$$\begin{aligned} x' &= g(t, x, y) + \frac{1}{n} y \\ y' &= h(t, x, y) + \frac{1}{n} x \end{aligned} \qquad (n = 1, 2, \ldots) \tag{13}$$

satisfy (11) and (12).

Suppose that

$$\varphi(\bar{t}) = 0 \qquad (t_0 \leqslant \bar{t} < T). \tag{14}$$

Denote by $(x_{1n}(t), y_{1n}(t))$, $(x_{2n}(t), y_{2n}(t))$ the solutions of (13) such that

$$x_{in}(\bar{t}) = x_i(\bar{t}), \qquad y_{in}(\bar{t}) = y_i(\bar{t}) \qquad (i = 1, 2;\ n = 1, 2, \ldots),$$

and by $\varphi_n(t) \overset{\text{df}}{=} (x_{1n}(t) - x_{2n}(t))(y_{1n}(t) - y_{2n}(t))$.

By (14) we get that $\varphi_n(\bar{t}) = 0$. It follows from the previous discussion that

$$\varphi_n(t) \geqslant 0 \qquad \text{for} \qquad t \geqslant \bar{t}. \tag{15}$$

On account of the almost uniform convergence of $g(t, x, y) + \frac{y}{n}$ and $h(t, x, y) + \frac{x}{n}$ to functions $g(t, x, y)$ and $h(t, x, y)$, respectively, and of the uniqueness property of (5) we conclude that

$$\varphi_n(t) \to \varphi(t).$$

This and (15) show that (I) holds also in the case of assumption c). Now it is sufficient to demonstrate that there exists number t^*, $t_0 \leqslant t^* < T$, such that the function $\varphi(t)$ is non-positive for $t^* \leqslant t < T$ or non-negative for $t^* \leqslant t < T$.

If $\varphi(t) < 0$ $(\varphi(t) > 0)$ for $t_0 < t < T$, then we put $t^* = t_0$. Suppose now that there is such t, $t_0 \leqslant \bar{t} < T$, that $\varphi(\bar{t}) = 0$. By (I) $\varphi(t) \geqslant 0$ for $\bar{t} \leqslant t < T$. In this case we put $t^* = \bar{t}$.

Thus, Lemma 2 is proved.

Notice that Lemma 2 together with Lemma 1 proves Theorem C.

Indeed, if $(x_0(t), y_0(t))$ for $t \geqslant 0$ is a bounded solution of (5), then it follows from Lemma 2 that the functions $x_0(t) - x_0(t+1)$, $y_0(t) - y_0(t+1)$ satisfy the assumptions of Lemma 1, but, as may easily be seen, Theorem C is a consequence of Lemma 1.

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MATHEMATICS

On Factor Groups of Some Subgroups of a Complete Direct Sum of Infinite Cyclic Groups

by

S. BALCERZYK

Presented by K. KURATOWSKI on January 20, 1959

We connect an Abelian group $S(\mathcal{B})$ with an arbitrary Boolean σ-algebra $\mathcal{B}$ (with unit V): elements of the group $S(\mathcal{B})$ are sequences

$$a = \langle a_n \rangle_{-\infty<n<+\infty} \tag{1}$$

satisfying the following conditions

$$a_n \in \mathcal{B} \quad \text{for all integers} \quad n \tag{2}$$

$$a_n \cap a_m = 0 \quad \text{for} \quad n \neq m \tag{3}$$

$$\bigcup_{-\infty<u<\infty} a_n = V \tag{4}$$

the sum $c = a + b$ of two such sequences a, b is a sequence $c = \langle c_n \rangle$ defined by equality

$$c_n = \bigcup_{-\infty<j<+\infty} a_j \cap b_{n-j} \text{ *)}. \tag{5}$$

For $\mathcal{B} = 2^T$ (the class of all subsets of a set T) the group $S(2^T)$ is isomorphic with a complete direct sum of $\overline{\overline{T}}$ copies of infinite cyclic groups.

Let $\mathcal{J}$ be an arbitrary finitely additive ideal in the algebra $\mathcal{B}$ and $\mathcal{J}_\beta$ the least σ-ideal containing $\mathcal{J}$. We define $S(\mathcal{J})$ as being a subgroup of all sequences $a = \langle a_n \rangle$ with $\bigcup_{\substack{-\infty<n<\infty \\ n\neq 0}} a_n \in \mathcal{J}$.

Let us repeat the definition of an algebraically compact group (see [2] p. 56): a group G is algebraically compact if and only if it is of the form $G = G_0 + \sum_p^* G_p$ and G_0 is a divisible group, G_p (for every prime) has no

*) The notion of group $S(\mathcal{B})$ is due to J. Łoś.

element of infinite p-height and is complete in its p-adic topology. The following theorem holds:

THEOREM 1. *If $\mathcal{J}$ is a finitely additive ideal in a Boolean σ-algebra $\mathcal{B}$, then the factor group $S(\mathcal{J}_\beta)/S(\mathcal{J})$ is a torsion-free algebraically compact group.*

In a special case of Boolean σ-algebra 2^T we shall consider more precisely subgroups $S(\mathcal{J}^\varkappa)$, where $\mathcal{J}^\varkappa$ is the ideal of all subsets of T of cardinality $< \aleph_\varkappa$. The only interesting case is that of $\mathcal{J}^\varkappa_\beta \neq \mathcal{J}^\varkappa$. It holds evidently if and only if $\aleph_\varkappa$ is a sum of a countable number of alephs less than $\aleph^\varkappa$, i. e. $\operatorname{cf}\varkappa = 0$ *). In that case $\mathcal{J}^\varkappa_\beta$ is identical with $\mathcal{J}^{\varkappa+1}$

Admitting Cantor's Generalized Continuum Hypothesis (i. e. $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every ordinal α) the following theorem may be proved:

THEOREM 2. *Let $S^*(T)$ be the complete direct sum of $\overline{T} = \aleph_\lambda$ copies of an infinite cyclic group, $\varkappa$ an ordinal with $\varkappa \leqslant \lambda$ and $\operatorname{cf}\varkappa = 0$. Then the factor group $S(\mathcal{J}^{\varkappa+1})/S(\mathcal{J}^\varkappa)$ is of the form:*

$$R^{\aleph_\lambda} + \sum_p{}^{*} \mathcal{J}_p^{\aleph^*_{\lambda-1}} \quad \textit{in case } \lambda \textit{ is not a limit ordinal}$$

$$R^{\aleph_\lambda} + \sum_p{}^{*} \operatorname{compl}(\mathcal{J}_p^{\aleph_\lambda}) \quad \textit{in case } \lambda \textit{ is a limit ordinal and } \operatorname{cf}\lambda > \varkappa$$

$$R^{\aleph_{\lambda+1}} + \sum_p{}^{*} \mathcal{J}_p^{\aleph^*_\lambda} \quad \textit{in case } \lambda \textit{ is a limit ordinal and } \operatorname{cf}\lambda \leqslant \varkappa \text{ **)}$$

Using Theorem 1 and the result of [1] we get easily:

THEOREM 3. *Let $\varkappa, \lambda$ be ordinals with $\varkappa \leqslant \lambda$ and $\operatorname{cf}\varkappa = 0$ and $S^*(T)$ be the complete direct sum of $\overline{T} = \aleph_\lambda$ copies of an infinite cyclic group. Then the group $S(\mathcal{J}^{\varkappa+1})/S(\mathcal{J}^\varkappa)$ is a direct summand of $S^*(T)/S(\mathcal{J}^\varkappa)$ and $S^*(T)/S(\mathcal{J}^\varkappa) \simeq S(\mathcal{J}^{\varkappa+1})/S(\mathcal{J}^\varkappa) + S^*(T)/S(\mathcal{J}^{\varkappa+1})$.*

The proofs of the above theorems will be published in Fundamenta Mathematicae.

I wish to express my gratitude and thanks to Professor J. Łoś and Mr. E. Sąsiada for their helpful suggestions.

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*) For an arbitrary ordinal α, $\operatorname{cf}\alpha$ is the least ordinal γ such that ω_α is confinal with ω_γ.

**) I_p is a group of p-adic integers, R that of rational numbers, $H^{\aleph^*_\alpha}(H^{\aleph_\alpha})$ is a complete (discrete) direct sum of $\aleph_\alpha$ copies of group H, $\operatorname{compl}(H)$ is a completion of group H in its p-adic topology.

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MATHEMATICS

Proof that Every Countable and Reduced Torsion-Free Abelian Group is Slender

by

E. SĄSIADA

Presented by K. KURATOWSKI, on January 19, 1959

We call a torsion-free Abelian group H *slender* *) if every homomorphism of a complete direct sum of countable sets of infinite cyclic groups $\{e_\nu\}(\nu = 1, 2, ...)$ into H sends almost all components $\{e_\nu\}$ into the zero of H.

The purpose of this note is to prove the following

THEOREM. *Every countable and reduced torsion-free Abelian group is slender.*

Proof. Let H be an arbitrary and reduced torsion-free group, and $S^* = \sum_{\nu=1}^{\infty}{}^{*} \{e_\nu\}$ the complete direct sum of infinite cyclic groups $\{e_\nu\}$ $(\nu = 1, 2, ...)$.

Suppose that H is not slender, i. e. there exists a homomorphism $h: S^* \to H$ such that, for infinitely many indexes ν, we have $h(e_\nu) \neq 0$. We can assume that

$$h(e_\nu) \neq 0 \quad \text{for every } \nu = 1, 2, ... \tag{1}$$

Since H is a reduced group, then

$$\bigcap_{n=1}^{\infty} nH = \{0\}. \tag{2}$$

It is easy to demonstrate from (1) and (2) that there exists an increasing sequence of positive integers $n_1 < n_2 < ...$ such that

$$h(n_\nu ! e_\nu) \text{ non } \epsilon\, n_{\nu+1} H \quad (\nu = 1, 2, ...). \tag{3}$$

*) This important notion in the theory of complete direct sums is due to J. Łoś (Cf [1]).

It follows from (3) that

(4) If $y = \langle y_1, \dots y_\nu, \dots\rangle$ is a non-zero element of S^* such that y_ν is equal to 0 or $\pm n_\nu! e_\nu (\nu = 1, 2, \dots)$, then $h(y) \neq 0$.

In fact: suppose that $h(y) = 0$ and let y_ν be the first component of y different from 0. Then $h(y_\nu) = \pm h(n_\nu! e_\nu) = h(\langle 0, \dots, 0, y_{\nu+1}, \dots\rangle)$, and this contradicts (3), since evidently $h(\langle 0, \dots 0, y_{\nu+1}, \dots\rangle) \in n_{\nu+1} H$.

Let now X be the set of all elements $x = \langle x_1, \dots x_\nu, \dots\rangle \in S^*$ such that x_ν be equal to 0 or $n_\nu! e_\nu$ $(\nu = 1, 2, \dots)$. The power of the set X is equal to the power of continuum. Since H is a countable there exist two different elements x_1 and x_2 of X such that $h(x_1) = h(x_2)$. Take $y = x_1 - x_2$. Then $h(y) = 0$. On the other hand, element y satisfies the assumptions of (4) and, therefore, $h(y) \neq 0$. The obtained contradiction proves that H must be slender.

Remark. Observe that the only use made of the assumption of countability of the group H was to demonstrate that the power of H is smaller than that of continuum. Therefore, we have:

Every reduced torsion-free group of a power smaller than that of continuum is slender.

The author wishes to thank Professor J. Łoś for advice in the preparation of this paper.

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MATHEMATICS

On the Isomorphism of Decompositions of Torsion-Free Abelian Groups into Complete Directs Sums of Groups of Rank One

by

E. SĄSIADA

Presented by K. KURATOWSKI, on January 19, 1959

We term isomorphic the two decompositions $G = \sum_{t \in T}^{*} G_t$ and $G = \sum_{t' \in T'}^{*} H_{t'}$ of the group G into the complete direct sums of groups $G_t (t \in T)$ and $H_{t'} (t' \in T')$, respectively, if there exists a one to one correspondence between the elements of T and T' such, that the components of G with corresponding indices are isomorphic.

It is an open question:

(D.1) *are any two decompositions of a torsion-free group into a complete direct sum of groups of rank isomorphic?*

It is easy to see that, for divisible groups, this question is equivalent to the question: Does $2^n = 2^m$ imply equality $n = m$ for every cardinal of n and m? For this reason the problem cannot be solved in general without the aid of a special set of theoretical assumptions.

Therefore A. G. Kurosh raised the following problem in connection with (D.1):

(D.2) *are any two canonical decompositions of a torsion-free group G isomorphic?*

A canonical decomposition of G is a decomposition $G = \sum_{\tau \in \Omega}^{*} G^{(\tau)}$ (generally non-unique), where $G^{(\tau)} (\tau \in \Omega)$ is the complete direct sum of groups of rank 1 of the same type τ, and Ω — the set of all types.

In this paper we give a positive answer to question (D.2.)*) (see Theorem 1). and, using this result, we prove that, in the most interesting

*) A partial result concerning the question mentioned was presented by A. P. Mishina [2].

case of reduced groups, the answer to question (D.1.) is also positive if the power of G is less than the first cardinal $\aleph_i$ of non-zero measure *) (see Theorem 2).

Lemmas

The following theorem of J. Łoś will play an essential role in the proof of the theorems announced:

(1) (Fuchs [1]) *If* $G = \sum_{t \in T}^{*} G_t$ *is the complete direct sum of arbitrary torsion-free Abelian groups* $G_t (t \in T)$ *and if the power of the set* T *is less than the first cardinal of non-zero measure, then, for every slender group* L, *we have:*

(i) *if* h *is a homomorphism of* G *into* L, *then* $h(G_t) \neq \{0\}$ *only for a finite set of indices* $t \in T$;

(ii) *if* h *is a homomorphism of* G *into* L, *then* $h(G_t) = \{0\}$ *for every* $t \in T$, *then* h *is a zero homomorphism;*

(iii) *the group of all homomorphisms of* G *into* L, $\mathrm{Hom}(G, L)$ *is isomorphic to the discrete direct sum* $\sum_{t \in T} \mathrm{Hom}(G_t, L)$ *of groups* $\mathrm{Hom}(G_t, L) (t \in T)$.

We call a torsion-free Abelian group L slender if every homomorphism of a complete direct sum of countable sets of infinite cyclic groups $\{e_\nu\}$ $(\nu = 1, 2, \ldots)$ into L sends almost all components $\{e_\nu\}$ into the zero of L.

(2) (Sąsiada [3]). *Every countable and reduced torsion-free Abelian group is slender.*

Now we shall prove

LEMMA 1. *Let* $G^{(\tau_0)}$ *be a complete direct sum of groups of rank* 1 *of the same type* τ_0. *If* h *is a non-zero homomorphism of* $G^{(\tau_0)}$ *into* $R^{(\tau)}$ *of rank* 1 *and of type* τ, *then* $\tau \geqslant \tau_0$.

Proof. We can assume that $R^{(\tau)}$ is a reduced group because otherwise Lemma 1 is obvious. Then, by (2), the group $R^{(\tau)}$ is slender. Let $G^{(\tau_0)} = \sum_{t \in T}^{*} R_t^{(\tau_0)}$ and let $x = (x_t)_{t \in T}$ be an element of $G^{(\tau_0)}$ such that $h(x) \neq 0$. Take for every characteristic $\chi \in \tau_0$ the subset T_χ of all $t \in T$ such that $x_t \neq 0$ and the characteristic of x_t be equal to χ, and let T_0 be the set of all $t \in T$ for which $x_t = 0$. Then the group $G^{(\tau_0)}$ can be represented in the form

*) Thus $\aleph_i$ is the least cardinal with the following property: There exists a σ-measure μ on the field of all subsets of a set T of power $\aleph_i$ such that μ takes values 0 and 1 only; it is 0 on the finite subsets of T and 1 for the whole set T. $\aleph_i$ is not less than the first aleph inaccessible in a strict sense (S. Ulam [4]) hence for example all cardinals $\aleph_0$, $2^{\aleph_0}$, $2^{2^{\aleph_0}}$, ... are less than $\aleph_i$.

(3) $G^{(\tau_0)} = G_0 + \sum_{\chi \in \tau_0}^{*} G_\chi$ where $G_0 = \sum_{t \in T_0}^{*} R_t$ and $G_\chi = \sum_{t \in T_\chi}^{*} R_t$ for every $\chi \in \tau_0$. In the decomposition (3) of $G^{(\tau_0)}$ the element $x = (x_t)$ takes the form $x = x^{(0)} + (x^{(\chi)})_{\chi \in T}$, where $x^{(0)} = 0$ and $x^{(\chi)} = (x_t)_{t \in T_\chi}$.

Since the power of the set of all characteristics $\chi \in \tau_0$ is $\leqslant \aleph_0$ and $R^{(\tau)}$ is slender, then in the face of (1) there exists a $\chi \in \tau_0$ such that $h(x^{(\chi)}) \neq 0$. Further, the element $x^{(\chi)}$ is of type τ_0 in the group $G^{(\tau_0)}$); thereby the type of $h(x^{(\chi)})$ which is equal to τ must be $\geqslant \tau_0$.

From Lemma 1 follows

LEMMA 2. *Let G be the complete direct sum of groups $R_t (t \in T)$ of rank 1 and let τ_0 be a fixed type such that the type of every $R_t (t \in R)$ be strictly greater than τ_0. Then the only homomorphism h of G into the group $R^{(\tau_0)}$ of rank 1 and of type τ_0 is a zero homomorphism.*

Proof By our assumption, the group $R^{(\tau_0)}$ is a reduced group and consequently slender. Take the canonical decomposition $G = \sum_{\tau \in \Omega}^{*} G^{(\tau)}$ of group G. By Lemma 1 the image $h(G^{(\tau)})$ of each $G^{(\tau)} (\tau \in \Omega)$ is $\{0\}$. Since the set Ω is of power $2^{\aleph_0}$ and $R^{(\tau)}$ is a slender group, hence, by applying Theorem (1) of J. Łoś to the decomposition $G = \sum_{\tau \in \Omega}^{*} G^{(\tau)}$, we find that the homomorphism h is a zero homomorphism.

Solution of Problem (D. 2.)

We shall prove the following

THEOREM 1. *If G and H are two isomorphic torsion-free Abelian groups and $G = \sum_{\tau \in \Omega}^{*} G^{(\tau)}$, $H = \sum_{\tau \in \Omega}^{*} H^{(\tau)}$ the canonical decompositions of G and H, respectively, then $G^{(\tau)}$ is isomorphic with $H^{(\tau)}$ for every $\tau \in \Omega$.*

Proof *) For every $\tau_0 \in \Omega$ both groups G and H can be decomposed into the direct sum of the form

(4) $G = G^{(\tau_0)} + A$ and $H = H^{(\tau_0)} + B$, where $A = \sum_{\tau \neq \tau_0}^{*} G^{(\tau)}$ and $B = \sum_{\tau \neq \tau_0}^{*} H^{(\tau)}$.

By assumption the group G is isomorphic to H. Let φ be this isomorphism.

We put

(5) $\varphi(G^{(\tau_0)}) = \bar{G}^{(\tau_0)}$, $\varphi(A) = \bar{A}$, $\varphi^{-1}(H^{(\tau_0)}) = \bar{H}^{(\tau_0)}$ and $\varphi^{-1}(B) = \bar{B}$.

Then we have

(6) $G = \bar{H}^{(\tau_0)} + \bar{B}$ and $H = \bar{G}^{(\tau_0)} + \bar{A}$.

From Lemma 1 it follows that

(7) if element $x = (x^{(\tau)})_{\tau \in \Omega}$ of G (of H) belongs to $\bar{H}^{(\tau_0)}$ (to $\bar{G}^{(\tau_0)}$) and a component $x^{(\tau)}$ is different from 0 then $\tau \geqslant \tau_0$.

*) The idea of this proof is the same as that of A. P. Mishina in [2].

In fact: since $G^{(\tau)} = \sum_{t \in T_\tau}^{*} R_t^{(\tau)}$ and $x^{(\tau)} = (x_t^{(\tau)})_{t \in T_\tau}$ are different from 0, then taking the projection onto $R_t^{(\tau)}$, for which $x_t^{(\tau)} \neq 0$, we obtain a non-zero homomorphism of $\bar{H}^{(\tau_0)}$ into $R_t^{(\tau)}$. It follows from Lemma 1 that $\tau \geqslant \tau_0$, and this completes the proof of (7).

Consider the projection π (with respect to the decomposition (4)) of $\bar{H}^{(\tau_0)}$ to $G^{(\tau_0)}$. We shall show that π is an isomorphism of $\bar{H}^{(\tau_0)}$ into $G^{(\tau_0)}$. Let $x = (x^{(\tau)})_{\tau \in \Omega}$ be an arbitrary element of $\bar{H}^{(\tau_0)}$ different from 0 and suppose that $\pi(x) = x^{(\tau_0)} = 0$. Then, by (7), for every component $x^{(\tau)}$ of x, we have

(8) $$\text{if} \quad x^{(\tau)} \neq 0, \quad \text{then} \quad \tau > \tau_0.$$

Let Ω' be the set of all $\tau \in \Omega$ such that $\tau > \tau_0$, and let $G' = \sum_{\tau \in \Omega'}^{*} G^{(\tau)} \subset G$. Then, in virtue of Lemma 2, we find that the projection of G' into $\bar{H}^{(\tau_0)}$ is $\{0\}$ which means that $G' \subset \bar{B}$. Since, by (8), the element x belongs to G', we have $x \in \bar{B}$ which, in the face of (6), contradicts the assumption that $x \in \bar{H}^{(\tau_0)}$ and $x \neq 0$.

Observe that it also follows from the symmetry of the assumptions that the projection $\bar{\pi}$ of $\bar{G}^{(\tau_0)}$ into $H^{(\tau_0)}$ is an isomorphism.

Now we shall show that π is an isomorphism onto $G^{(\tau_0)}$. Let x_0 be an arbitrary element of $G^{(\tau_0)}$. Then, element $\varphi(x_0) \in \bar{G}^{(\tau_0)}$ and can be represented (in the decomposition (4) of H) as the sum $\varphi(x_0) = \bar{\pi}\varphi(x_0) - b$, where $\bar{\pi}\varphi(x_0) \in H^{(\tau_0)}$ and $b \in B$. From (7) and from Lemma 2 it is easy to obtain that b belongs to $\bar{A}$. Observe that, by (5), $\varphi^{-1}\bar{\pi}\varphi(x_0) = \varphi^{-1}\varphi(x_0) + \varphi^{-1}(b) = x_0 + \varphi^{-1}(b)$ belongs to $\bar{H}^{(\tau_0)}$. Since $b \in \bar{A}$, then $\varphi^{-1}(b) \in A$ and consequently $\pi\varphi^{-1}(b) = 0$. From this it follows that $x_0 = \pi(\varphi^{-1}\bar{\pi}\varphi(x_0))$ and $\varphi^{-1}\bar{\pi}\varphi(x_0) \in \bar{H}^{(\tau_0)}$ and this completes the proof of our theorem.

Partial solution of problem (D. 1.)

We shall prove the following

THEOREM 2. *If the power of a reduced torsion-free Abelian group G is less than the first cardinal of non-zero measure, then any two decompositions of G into the complete direct sum of groups of rank 1 are isomorphic.*

Proof. Let $G = \sum_{t \in T}^{*} R_t$ and $G = \sum_{t' \in T'}^{*} R_{t'}$ be two such decompositions of G. Take the canonical decompositions $G = \sum_{\tau \in \Omega}^{*} G^{(\tau)}$ and $G = \sum_{\tau \in \Omega}^{*} H^{(\tau)}$, where $G^{(\tau)} = \sum_{t \in T_\tau}^{*} R_t$ and $H^{(\tau)} = \sum_{t' \in T'_\tau}^{*} R_{t'}$.

From Theorem 1, it follows that $G^{(\tau)}$ is isomorphic to $H^{(\tau)}$ for each $\tau \in \Omega$. It remains to be proved that the decompositions $G^{(\tau)} = \sum_{t \in T_\tau}^{*} R_t$ and $H^{(\tau)} = \sum_{t' \in T'_\tau}^{*} R_{t'}$ are isomorphic. Since all the $R_t (t \in T_\tau)$ and also

$R_{t'}(t' \in T'_\tau)$ are isomorphic we must only show that the power of T_τ is equal to the power of T'_τ.

Let us consider the group $R^{(\tau)}$ of rank 1 and of type τ. Then all the groups of homomorphisms $\mathrm{Hom}(R_t, R^{(\tau)})(t \in T_\tau)$ and $\mathrm{Hom}(R_{t'}, R^{(\tau)})(t' \in T'_\tau)$ are isomorphic torsion-free groups of rank 1. Since $G^{(\tau)}$ is isomorphic to $H^{(\tau)}$ them, from the assumption of the cardinality of G and from Theorem (1) of J. Łoś, it follows that the discrete direct sums $\sum_{t \in T_\tau} \mathrm{Hom}(R_t, R^{(\tau)})$ and $\sum_{t' \in T'_\tau} \mathrm{Hom}(R_{t'}, R^{(\tau)})$ are isomorphic, thus the powers of T_τ and T'_τ must be equal.

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MATHEMATIK

Sphärische Distributionen auf homogenen Räumen. Eine Verallgemeinerung des Satzes von Bochner-Weil-Rajkow

von

K. MAURIN

Vorgelegt von S. MAZUR am 30. Januar, 1959

In dieser Note geben wir einige Anwendungen des von uns in [6] bewiesenen Satzes über die Entwicklung der positiv definiter Kerne nach Eigendistributionen. Diese Anwendungen umfassen erstens einen Satz von Gelfand über die Entwicklung positiv definiter invarianter Kerne nach (verallgemeinerten) sphärischen Funktionen und zweitens verallgemeinern und verschärfen den Satz von Bochner-A. Weil und Rajkow*).

Es sei (Ω_n, G) ein homogener Raum, d. h. Ω_n ist eine n-dimensionale differenzierbare Mannigfaltigkeit und G die transitive Liesche Gruppe der Abbildungen von Ω_n:

$$\Omega_n \ni x \to gx \in \Omega_n .$$

Die Gruppe G ist separabel und zusammenhängend. Es sei $C_0^\infty(\Omega_n)$ die Menge der beliebig oft differenzierbaren Funktionen mit kompakten Trägern in Ω_n. Es sei $\Theta(\varphi, \psi)$ eine positiv definite invariante hermitische Form auf $C_0^\infty(\Omega_n)$, d. h.

$$\Theta(L_g\varphi, L_g\psi) \equiv \Theta(\varphi, \psi) \quad \text{für} \quad g \in G;\ \varphi, \psi \in C_0^\infty(\Omega_n), \tag{1}$$

wobei

$$L_g\varphi(x) \overset{\text{df}}{=} \varphi(g^{-1}x) .$$

$\Theta(x, y)$ sei ein Kern — im Sinne von L. Schwartz — m-ter Ordnung auf Ω_n, $m = 0, 1, 2, \ldots, \infty$; d. h. die bilineare Form $\Theta(\cdot, \cdot)$ ist stetig auf $\mathcal{D}^m(\Omega_n) \times \mathcal{D}^m(\Omega_n)$. Durch die Vervollständigung von $C_0^\infty(\Omega_n)$ erhalten wir den Hilbertschen Raum H mit dem Skalarprodukt $\Theta(\varphi, \psi)$.

*) Eine Anwendung zur Quantenelektrodynamik soll bald folgen.

Es sei schliesslich $e_1, \ldots, e_N$ eine Basis der Lieschen Algebra $\mathfrak{g}$ von G und $C_{\lambda\mu}^{\varkappa}$ die Strukturkonstanten von G, d. h.

$$[e_i, e_l] = \sum_{\varkappa=1}^{N} C_{il}^{\varkappa} e_{\varkappa}.$$

Die assoziative Algebra $\mathfrak{B}$ mit Generatoren $e_1, \ldots, e_N$ und definierenden Relationen $e_i e_l = e_l e_i + \sum_{\varkappa=1}^{N} C_{il}^{\varkappa} e_{\varkappa}$ heisst *universelle einhüllende Algebra* von $\mathfrak{g}$. J. M. Gelfand [3] hat das Zentrum $\mathfrak{Z}$ von $\mathfrak{B}$ folgendermassen bestimmt: das Polynom

$$(2) \qquad P_\nu(e_1, \ldots, e_N) = a_\nu 1 + \sum_i a_\nu^i e_i + \sum_{i,k}{}' a_\nu^{ik} e_i e_\varkappa + \sum_{i,k,l} a_\nu^{ikl} e_i e_k e_l + \ldots$$

(die Koeffizienten $a_\nu^{ik}, a_\nu^{ikl}, \ldots$ sind symmetrisch) ist mit allen e_k vertauschbar ($\in \mathfrak{Z}$) dann (und nur dann), wenn die Formen $\sum a_\nu^i \xi_i$, $\sum a_\nu^{ik} \xi_i \xi_k, \ldots$ Invarianten der zugeordneten Gruppe von G sind.

Aus (1) folgt es, dass $G \ni g \to L_g \in \mathcal{L}(H)$ *) eine unitäre Darstellung der Lieschen Gruppe G im Darstellungsraum H ist. Zugleich werden die Generatoren $e_1, \ldots, e_N$ durch die Lieschen Differentialoperatoren $X_1, \ldots, X_N$ dargestellt (X_k bedeutet die Differentiation in der Richtung der 1-gliedrigen Untergruppe $\exp(te_k)$). Die Operatoren X_k sind auf der in H dichten linearen Untermenge — dem sgn. Gärdingschen Raum — erklärt. Nach einem Satz von J. E. Segal [7] sind die *symmetrischen* Polynome

$$(3) \qquad \Delta_\nu \overset{\text{df}}{=} P_\nu(iX_1, \ldots, iX_N), \quad P_\nu \in \mathfrak{Z}$$

wesentlich selbstadjungiert (d. h. ihre Abschliessung Δ_ν^* ist selbst adjungiert: $\Delta_\nu^{**} = \Delta_\nu^*$. Wir bezeichnen die selbstadjungierte Fortsetzung Δ_ν^* von Δ_ν wieder mit Δ_ν) und mit allen L_g vertauschbar.

Jetzt wenden wir den von uns in [6] bewiesenen Satz auf die durch die Operatoren Δ_ν erzeugte Algebra $\mathcal{A}$ folgendermassen an:

Durch die Cayleysche Transformation $\mathcal{C}_{\Delta_\nu}$ von Δ_ν erhalten wir die vertauschbaren unitären Operatoren

$$(4) \qquad U_k = \mathcal{C}_{\Delta_\nu} \overset{\text{df}}{=} (\Delta_\nu - iI)(\Delta_\nu + I)^{-1}.$$

Auf der Menge $\{U_\nu\}$ spannen wir die schwach abgeschlossene von Neumannsche Algebra (die sg. W^*-Algebra) $\mathcal{A}$ auf. Die Algebra $\mathcal{A}$ erzeugt bekanntlich ein direktes Integral

$$(5) \qquad \mathfrak{H} = \int_\Lambda \oplus \mathfrak{H}(\lambda) d\mu(\lambda)$$

*) $\mathcal{L}(H)$ bedeutet die Banach-Algebra der beschränkten linearen Operatoren in H.

und eine unitäre Abbildung F – die sgn. Fouriersche Transformation $H \xrightarrow{F} \mathfrak{H}$, die die Algebra $\mathcal{A}$ diagonalisiert (für Definitionen vgl. [6]):

$$\mathcal{L}(H) \supset \mathcal{A} \ni A_f \leftrightarrow \hat{A}_f \in \hat{\mathcal{A}} \subset \mathcal{L}(\mathfrak{H}), \tag{6}$$

$$FA_fF^{-1} = \hat{A}_f. \tag{7}$$

$\hat{A}_f$ sind die diagonalen Operatoren: $\hat{A}_f = (f(\lambda)I(\lambda))$, wo $I(\lambda)$ der Identitätsoperator im Hilbertschen Raume $\mathfrak{H}(\lambda)$, $\lambda \in \Lambda$, ist.

$$L^\infty(\Lambda) \ni f \leftrightarrow A_f \in \mathcal{A}$$

ist der Gelfandsche Isomorphismus. Die Fourier-Transformation F diagonalisiert auch die (unbeschränkten) Laplaceschen Operatoren Δ_ν *) folgendermassen: falls $U_\nu = \mathcal{C}_{\Delta_\nu} = A_{f_\nu}$, dann

$$F\Delta_\nu F^{-1} = \hat{A}_{\Delta_\nu(\cdot)} = \big(\Delta_\nu(\lambda)I(\lambda)\big) \tag{8}$$

wo die Funktion

$$\lambda \to \Delta_\nu(\lambda) = i\frac{1+f_\nu(\lambda)}{1-f_\nu(\lambda)} \tag{9}$$

μ-messbar und μ-f. ü. endlich ist.

Da die unitären Operatoren L_g mit $\mathcal{A}$ vertauschbar sind, haben wir die folgende faktorielle Zerlegung der Darstellung

$$\mathcal{L}(\mathfrak{H}) \ni \hat{L}_g \overset{\text{df}}{=} FL_gF^{-1} = \int_\Lambda \oplus L_g(\lambda)\,d\mu(\lambda). \tag{10}$$

Wir setzen voraus, dass die unitären Darstellungen

$$G \ni g \to L_g(\lambda) \in \mathcal{L}\big(\mathfrak{H}(\lambda)\big) \tag{11}$$

faktoriell sind, d. h. die auf den $(L_g(\lambda))$ aufgespannte W^* Algebra $\mathcal{F}(\lambda)$ ist Faktor, d. h. das Zentrum $\mathcal{F}(\lambda) \cap \mathcal{F}(\lambda)'$ von $\mathcal{F}(\lambda)$ besteht aus den skalaren Operatoren $aI(\lambda)$, $a \in \mathcal{C}^1$.

Es gilt also der folgende

Satz 1. *Der positiv definite invariante Kern $\Theta(\cdot,\cdot)$ m-ter Ordnung auf (Ω_n, G) induziert eine Zerlegung des Hilbertschen Raumes H in das direkte Integral $\mathfrak{H}$ (5). Dabei diagonalisiert die Fouriersche Transformation* (6), (7) *die Laplaceschen Operatoren Δ_ν auf (Ω_n, G)* – (8). *Der Kern $\Theta(x,y)$ wird nach den sphärischen Kernen $\Theta(x, y; \lambda)$ entwickelt:*

$$\Theta(x, y) = \int_\Lambda \Theta(x, y; \lambda)\,d\mu(\lambda), \tag{12}$$

*) Diese Bezeichnung stammt von Gelfand [2]. In diesem Zusammenhang vgl. auch die grundlegende Abhandlung von F. A. Berezin [1]. Falls $\Omega_2 = S_2 = (S_2, \mathcal{C}_3^*)$ die 2-dimensionale Sphäre ist, sind tatsächlich die Δ_ν Polynome des klassischen Laplace-Operators auf S_2.

genauer:

$$\Theta(\varphi, \psi) = \int_{\Lambda} \Theta_\lambda(\varphi, \psi) d\mu(\lambda) \qquad \varphi, \psi \in C_0^\infty(\Omega_n)$$

(13)

$$\Theta_\lambda(\varphi, \psi) = \int_{\Omega_n \times \Omega_n} \Theta(x, y; \lambda) \varphi(x) \overline{\psi(y)} dx dy .$$

Die positiv definite Kerne $\Theta_\lambda(\cdot, \cdot)$ *sind*:

1° *invariant, d. h.*

$$\Theta_\lambda(L_g\varphi, L_g\psi) \equiv \Theta_\lambda(\varphi, \psi), \qquad g \in G \tag{14}$$

2° *Eigendistributionen der Laplace-Operatoren* (3):

$$\Delta_\nu^x \Theta(x, y; \lambda) = \overline{\Delta}_\nu^y \Theta(x, y; \lambda) = \Delta_\nu(\lambda) \Theta(x, y; \lambda) \tag{15}$$

Die Eigenwerte $\Delta_\nu(\lambda)$, *sind durch die Relationen* (9) *definiert. Die Ordnung von* $\Theta_\lambda(\cdot, \cdot)$ *ist* $\leqslant m + n$ $(m = 0, 1, \ldots, \infty)$.

Die durch den Kern Θ *induzierte unitäre Darstellung*

$$G \ni g \to L_g \in \mathcal{L}(H)$$

der separablen zusammenhängenden Lieschen Gruppe G *zerfällt in die faktoriellen Darstellungen* (10).

Bemerkung. Man kann die Ordnung der Kerne $\Theta_\lambda(\cdot, \cdot)$ sogar auf $m + \left[\frac{n}{2}\right] + 1$ herunterdrücken, was an einer anderen Stelle bewiesen werden soll.

Wenn wir insbesondere als Ω_n die Gruppenmannigfaltigkeit G nehmen, erhalten wir die folgende Verallgemeinerung und Verschärfung der Sätze von S. Bochner ($G = \boldsymbol{E}^1$), A. Weil-Rajkow (G-abelsch), H. Weyl (G-kompakt):

Satz 2. *Es sei* $\Theta(\cdot, \cdot)$ *der positiv definite invariante Kern* m*-ter Ordnung — also eine Spur in der Terminologie von R. Godement* [5] *— auf der zusammenhängenden separablen Lieschen Gruppe* G. *Dann induziert* Θ *eine "reguläre" unitare Darstellung von* G

$$G \ni g \to L_g \in \mathcal{L}(H),$$

die in faktorielle Darstellungen $g \to L_g(\lambda) \in \mathcal{L}(\mathfrak{H}(\lambda))$ *zerfällt. Die Spur* $\Theta(x, y)$ *lässt sich nach den Charakteren* $\Theta_\lambda(x, y)$ *der Darstellungen* $L_g(\lambda)$ *entwickeln, wobei die Relationen* (11)-(15) *gelten. Die Charaktere* $\Theta_\lambda(\cdot, \cdot)$ *sind also die Eigendistributionnen* $\left(\text{der Ordnung } \leqslant m + \frac{N}{2} + 1\right)$ *der Laplaceschen Operatoren* Δ_ν *auf* G.

Bemerkung. Wie schon R. Godement in [4] bemerkt hat führen die *stetigen* Charaktere blos zur faktoriellen Darstellungen *endlicher* Klasse, was offenbar eine unsachgemässe Einschränkung ist. Deshalb hat Gode-

ment im Falle der unimodulären Lieschen Gruppe die Definition der Charaktere (vgl. [5], S. 83) eingeführt, die wir hier aufgenommen haben.

Die von uns angenommene Definition bildet also eine sinngemässe Fortsetzung der Definition von Godement auf beliebige (zusammenhängende, separable) Liesche Gruppen.

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Relations Between the Polynomial Method and the Factorization Method

by

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Introduction

The aim of the present paper is to find connections between two methods of solving eigenvalue problems, namely between the polynomial method in the form given by Rubinowicz [1] and the factorization method in Infeld's form [2], for the finite interval problem.

Starting from the first order operators for the P-functions, given by Inui [3] we show that the sets of functions P_m ($m = 1, 2, \ldots$) connected by single Inui operators are eigensolutions of problems which can be treated by the polynomial and the factorization method. On the other hand, the equations in which the eigenfunctions P_m and P_{m+1} are connected by an irreducible product of two Inui operators are in general not factorizable, though we can solve them by the polynomial method.

The Inui operators lead to all possible factorizations in a finite interval problem. It follows from this that all factorizable problems can be treated by the polynomial method but the reverse is not true.

1. The simplified polynomial method

We consider equations of the type

$$\frac{d}{d\xi}\left(p\frac{df}{d\xi}\right)-(q-\lambda\varrho)f=0\,, \tag{1.1}$$

which have, for eigenvalues $\lambda(n)$ of λ, solutions in the form

$$f_{mn}=(h\xi^{1-1/h}p)^{-1/2}P_{mn}; \qquad (h-\text{constant}) \tag{1.2}$$

m is here the separation constant and P_{mn} a P-function:

$$P_{mn}=P\left\{\begin{matrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma'\end{matrix};\,\xi\right\} \qquad \alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1 \tag{1.3}$$

Comparing (1.1), after inserting therein (1.2), with the P-equation, we get first the conditions for $\alpha, \alpha', \beta \ldots$:

$$\alpha+\alpha'=\frac{1}{h} \qquad \beta+\beta'=-\frac{1}{h} \qquad \gamma+\gamma'=1. \tag{1.4}$$

Then, P_{mn} has the form

$$P_{mn}=P\left\{\begin{matrix} \alpha & \beta & \gamma \\ 1/h-\alpha & -1/h-\beta & 1-\gamma \end{matrix}; \xi\right\} \tag{1.5}$$

and its equation becomes

$$\frac{d^2P}{d\xi^2}+\frac{1-1/h}{\xi}\frac{dP}{d\xi}+\left(\frac{-\gamma(\gamma-1)}{1-\xi}-\frac{\alpha(\alpha-1/h)}{\xi}+\beta\left(\beta+\frac{1}{h}\right)\right)\frac{1}{\xi(1-\xi)}P=0. \tag{1.6}$$

Comparing the coefficients of P in (1.1) and (1.6) we obtain the condition for the polynomial method

$$\left\{\frac{1}{2p}\frac{dp}{d\xi}\right\}^2-\frac{1}{2p}\frac{d^2p}{d\xi^2}-\frac{1-1/h}{\xi^2}-\frac{q-\lambda\varrho}{p} = \frac{1}{\xi(1-\xi)}\left[\frac{-\gamma(\gamma-1)}{1-\xi}-\frac{\alpha(\alpha-1/h)}{\xi}+\beta\left(\beta+\frac{1}{h}\right)\right]. \tag{1.7}$$

If p, q, ϱ, are known we can find α, β, γ. The eigenvalues $\lambda(n)$ of λ can be determined from the condition

$$\alpha+\beta+\gamma=-k \quad \text{or} \quad \alpha-\beta+\gamma-1/h=-k \quad (k-\text{positive integer}). \tag{1.8}$$

2. The factorization method

By a well known transformation of variables

$$u=P\cdot\left(\frac{\epsilon}{h\xi^{1-1/n}}\right)^{1/2}, \qquad v=\int \epsilon\, d\xi, \qquad \epsilon=\sqrt{\frac{\varrho}{p}} \tag{2.1}$$

we obtain from (1.1) an equation of the form

$$\frac{d^2u}{dv^2}+[r(v,m)+\lambda]u=0, \quad \text{where} \quad r(v,m)=\frac{1}{\sqrt[4]{\varrho p}}\frac{d^2\sqrt{\varrho p}}{dv^2}-\frac{q}{p}. \tag{2.2}$$

This equation is factorizable if it can be written in the form of a product of two first-order differential operators in two ways:

$$\left(k(v,m)-\frac{d}{dv}\right)\left(k(v,m)+\frac{d}{dv}\right)u_{mn}=[L(m)-\lambda]u_{mn}, \tag{2.3a}$$

$$\left(k(v,m+1)+\frac{d}{dv}\right)\left(k(v,m+1)-\frac{d}{dv}\right)u_{mn}=[L(m+1)-\lambda]u_{mn}. \tag{2.3b}$$

Each form of equation (2.3) is equivalent to two first-order recurrence relations

$$\text{(2.4a)} \qquad \left(k(v, m+1) - \frac{d}{dv}\right) u_{mn} = A_{m+1,n} u_{m+1,n}$$

$$A_{mn} = \sqrt{L(m) - \lambda(n)}$$

$$\text{(2.4b)} \qquad \left(k(v, m+1) + \frac{d}{dv}\right) u_{m+1,n} = A_{m+1,n} u_{mn}$$

which enable us to obtain different eigenfunctions u_{mn} for the same $\lambda(n)$ starting from an equation for which $A_{mn} = 0$. This last condition determines at the same time the eigenvalues of $\lambda(n)$.

Formulae (2.1) can be factorized if $k(v, m)$ satisfies the equation

$$\text{(2.5)} \qquad \frac{d}{dv}[k(v, m+1) + k(v, m)] + k^2(v, m+1) - k^2(v, m) = L(m) - L(m+1).$$

(the right-hand side depends only on m)

All factorizable equations are tabulated (cf. [2]). For the finite interval there are two types of factorization A and E.

3. Inui's recurrence formulae

Inui [3] considers operators of the form

$$\text{(3.1)} \qquad \xi^{C_1}(1-\xi)^{D_1} \frac{d}{d\xi} \xi^C (1-\xi)^D$$

which, acting on a P-function transform it into another P-function. We know that $\xi^C(1-\xi)^D P\left\{\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; \xi\right\} = P\left\{\begin{matrix} \alpha+C & \beta-C-D & \gamma+D \\ \alpha'+C & \beta'-C-D & \gamma'+D \end{matrix}; \xi\right\}$. On the other hand, the derivative $\frac{d}{d\xi} P\left\{\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{matrix}; \xi\right\} = Q\left\{\begin{matrix} \alpha-1 & \beta+1 & \gamma-1 \\ \alpha'-1 & \beta'+1 & \gamma'-1 \end{matrix}; \xi\right\}$ is, in general, not a P-function, because the condition (1.3) $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma' = 1$ is not satisfied. There are, however, 18 special cases when the differentiation of P gives another P-function. By a suitable choice of $\xi^C(1-\xi)^D$ we can obtain from an arbitrary P-function one of the 18 special cases. Thus, for any P-function there extist 18 operators $\frac{d}{d\xi}\xi^C(1-\xi)^D$ which transform it into their P-functions.

The operators $\frac{d}{d\xi}\xi^C(1-\xi)^D$ destroy the relation (1.4) which is a necessary condition for the simplified polynomial method. In order to maintain this relation we must multiply this operator by an adequately chosen factor $\xi^{C_1}(1-\xi)^{D_1}$. There exists only one such factor for each of the 18 Inui operators, listed in Table I. We see that the operators in this table relate mutually P-functions for which $|\alpha_0|+|\gamma_0|+|\beta_0|$ ($\alpha_0 = \alpha(m+1)-\alpha(m)\ldots$) is equal to 1.

4. Differential recurrence relations between two solutions obtained by the polynomial method

If α, β, γ of (1.6) satisfy the condition (1.7) they depend on λ, m and on the form of functions p, q, ϱ. Different equations (1.1) lead to different dependence of α, β, γ on λ and m and to different sets of solutions P_{mn}. (1.7) together with (1.8) lead to the condition that α, β, γ must be linear functions of m, and if we denote $\bar{\alpha} = \alpha, \beta, \gamma$, $\bar{\alpha}(m) = \bar{\alpha}_0 m + \bar{\alpha}_1$ we have

$$(4.1)\quad \alpha_0 = \frac{l_1}{2}, \qquad \beta_0 = \frac{l_2 - l_3}{2}, \qquad \gamma_0 = \frac{l_2 + l_3 - l_1}{2}; \; l_1, l_2, l_3 \quad \text{integers.}$$

We know that in this case there exist two recurrence relations between any two functions $P_m = P_{\bar{\alpha}}$ and $P_{m+1} = P_{\bar{\alpha}+\bar{\alpha}_0}$, because they are contiguous functions

$$(4.2\text{a})\quad H_{1\bar{\alpha}\bar{\alpha}_0} P_{\bar{\alpha}} = K_{1\bar{\alpha}\bar{\alpha}_0} P_{\bar{\alpha}+\bar{\alpha}_0}$$

$$(4.2\text{b})\quad H_{2\bar{\alpha}\bar{\alpha}_0} P_{\bar{\alpha}+\bar{\alpha}_0} = K_{2\bar{\alpha}\bar{\alpha}_0} P_{\bar{\alpha}}$$

$$H_{i\bar{\alpha}\bar{\alpha}_0} = A_{i\bar{\alpha}\bar{\alpha}_0}\left[(-1)^i B_{i\bar{\alpha}\bar{\alpha}_0} + \frac{d}{d\xi}\right] \qquad i = 1, 2.$$

$$K_{i\bar{\alpha}\bar{\alpha}_0} = \text{const}.$$

On the other hand, if (4.1) is satisfied we can connect $P_{\bar{\alpha}}$ and $P_{\bar{\alpha}+\bar{\alpha}_0}$ by a product of Inui operators:

$$(4.3\text{a})\quad \left(\prod_j O_{1j}\right) P_{\bar{\alpha}} = K_{1\bar{\alpha}\bar{\alpha}_0} P_{\bar{\alpha}+\bar{\alpha}_0}$$

$$(4.3\text{b})\quad \left(\prod_j O_{2j}\right) P_{\alpha+\alpha_0} = K_{2\bar{\alpha}\bar{\alpha}_0} P_{\bar{\alpha}} \qquad \text{where} \qquad O_{ij} = \xi^{C^*_{ij}}(1-\xi)^{D^*_{ij}} \frac{d}{d\xi} \xi^{C_{ij}}(1-\xi)^{D_{ij}}$$

$$i = 1, 2.$$

Using (1.6) we can reduce this product to one first order operator, and then, of course, we have $\prod_j O_{ij} = H_{i\bar{\alpha}\bar{\alpha}_0}$.

We now want to find conditions, which (4.2) must satisfy, if they are to connect two solutions of (1.6) with the same $\lambda(n)$. Multiplying the first equation (4.2a) by $H_{2\bar{\alpha}\bar{\alpha}_0}$ and the second (4.2b) by $H_{1\bar{\alpha}\bar{\alpha}_0}$ we obtain

$$(4.4\text{a})\quad P''_{\bar{\alpha}} + \left(B_{2\bar{\alpha}\bar{\alpha}_0} - B_{1\bar{\alpha}\bar{\alpha}_0} + \frac{d}{d\xi}\log A_{1\bar{\alpha}\bar{\alpha}_0}\right) P'_{\bar{\alpha}}$$
$$+ \left\{ - B_{1\bar{\alpha}\bar{\alpha}_0}\left(B_{2\bar{\alpha}\bar{\alpha}_0} + \frac{d}{d\xi}\log A_{1\bar{\alpha}\bar{\alpha}_0}\right) - B'_{1\bar{\alpha}\bar{\alpha}_0} - \frac{K_{1\bar{\alpha}\bar{\alpha}_0} K_{2\bar{\alpha}\bar{\alpha}_0}}{A_{1\bar{\alpha}\bar{\alpha}_0} A_{2\bar{\alpha}\bar{\alpha}_0}} \right\} P_{\bar{\alpha}} = 0$$

$$(4.4\text{b})\quad P''_{\bar{\alpha}+\bar{\alpha}_0} + \left(B_{2\bar{\alpha}\bar{\alpha}_0} - B_{1\bar{\alpha}\bar{\alpha}_0} + \frac{d}{d\xi}\log A_{2\bar{\alpha}\bar{\alpha}_0}\right) P'_{\bar{\alpha}+\bar{\alpha}_0}$$
$$+ \left\{ B_{2\bar{\alpha}\bar{\alpha}_0}\left(- B_{1\bar{\alpha}\bar{\alpha}_0} + \frac{d}{d\xi}\log A_{2\bar{\alpha}\bar{\alpha}_0}\right) + B'_{2\bar{\alpha}\bar{\alpha}_0} - \frac{K_{1\bar{\alpha}\bar{\alpha}_0} K_{2\bar{\alpha}\bar{\alpha}_0}}{A_{1\bar{\alpha}\bar{\alpha}_0} A_{2\bar{\alpha}\bar{\alpha}_0}} \right\} P_{\bar{\alpha}+\bar{\alpha}_0} = 0.$$

Comparing coefficients of P' with (1.6) we get

$$(4.5)\qquad B_{2\bar{\alpha}\bar{\alpha}_0} - B_{1\bar{\alpha}\bar{\alpha}_0} + \frac{d}{d\xi}\log A_{i\bar{\alpha}\bar{\alpha}_0} = \frac{1 - 1/h}{\xi}, \qquad i = 1, 2.$$

Thus, $A_{1\bar{\alpha}\bar{\alpha}_0} = C \cdot A_{2\bar{\alpha}\bar{\alpha}_0}$, and if $K_{1\bar{\alpha}\bar{\alpha}_0}, K_{2\bar{\alpha}\bar{\alpha}_0}$ are suitably chosen, then

$$-A_{1\bar{\alpha}\bar{\alpha}_0} = A_{2\bar{\alpha}\bar{\alpha}_0} = A_{\bar{\alpha}\bar{\alpha}_0}. \tag{4.5'}$$

Putting $m-1$ instead of m, we obtain from (4.4) the condition

$$B_{1\alpha\bar{\alpha}_0}\left(B_{1\bar{\alpha}\bar{\alpha}_0}+\frac{1-1/h}{\xi}\right)+B'_{1\bar{\alpha}\bar{\alpha}_0}-B_{2(\bar{\alpha}-\bar{\alpha}_0)\bar{\alpha}_0}\left(B_{2(\bar{\alpha}-\bar{\alpha}_0)\bar{\alpha}_0}-\frac{1-1/h}{\xi}\right) + B'_{2(\bar{\alpha}-\bar{\alpha}_0)\bar{\alpha}_0} = -\frac{K_{1\bar{\alpha}\bar{\alpha}_0}K_{2\bar{\alpha}\bar{\alpha}_0}}{A^2_{\bar{\alpha}\bar{\alpha}_0}}+\frac{K_{1(\bar{\alpha}-\bar{\alpha}_0)\bar{\alpha}_0}K_{2(\bar{\alpha}-\bar{\alpha}_0)\bar{\alpha}_0}}{A^2_{(\bar{\alpha}-\bar{\alpha}_0)\bar{\alpha}_0}} \tag{4.6}$$

that has to be fulfilled by $H_{i\bar{\alpha}\bar{\alpha}_0}$ which has now the form

$$H_{i\bar{\alpha}\bar{\alpha}_0} = A_{\bar{\alpha}\bar{\alpha}_0}\left(B_{i\bar{\alpha}\bar{\alpha}_0}-(-1)^n\frac{d}{d\xi}\right) \tag{4.7}$$

(4.4ab) become now (we write $K_{\bar{\alpha}\bar{\alpha}_0}$ instead of $K_{1\bar{\alpha}\bar{\alpha}_0}K_{2\bar{\alpha}\bar{\alpha}_0}$)

$$P''_{\bar{\alpha}}+\frac{1-1/h}{\xi}P'_{\bar{\alpha}}+\left\{-B_{1\bar{\alpha}\bar{\alpha}_0}\left(B_{1\bar{\alpha}\bar{\alpha}_0}+\frac{1-1/h}{\xi}\right)-B'_{1\bar{\alpha}\bar{\alpha}_0}-\frac{K_{\bar{\alpha}\bar{\alpha}_0}}{A^2_{\bar{\alpha}\bar{\alpha}_0}}\right\}P_{\bar{\alpha}}=0$$

(4.8ab)

$$P''_{\bar{\alpha}+\bar{\alpha}_0}+\frac{1-1/h}{\xi}P'_{\alpha+\alpha_0}+\left\{B_{2\bar{\alpha}\bar{\alpha}_0}\left(-B_{2\bar{\alpha}\bar{\alpha}_0}+\frac{1-1/h}{\xi}\right)+B'_{2\bar{\alpha}\bar{\alpha}_0}-\frac{K_{\bar{\alpha}\bar{\alpha}_0}}{A^2_{\bar{\alpha}\bar{\alpha}_0}}\right\}P_{\bar{\alpha}+\bar{\alpha}_0}=0.$$

Each of the 18 Inui operators from Table I satisfies, of course, all these conditions. These operators can be arranged in pairs relating two solutions of the same Eq. (1.6) $P_{mn} = P_{\bar{\alpha}}$ and $P_{m+1,n} = P_{\bar{\alpha}+\bar{\alpha}_0}$. For each pair of operators the value of $\bar{\alpha}_0$ and $\lambda(n)$, and also the form of (1.6) are completely determined, Thus, we have 9 pairs of operators corresponding to 9 types of Eqs. (1.6). Each type has determined eigenvalues $\lambda(n)$ and the ration $\frac{q}{p}, \frac{\varrho}{p}$ can be obtained for each type grom (1.7).

If $P_{\bar{\alpha}}$ is a solution of (1.6), then $P_{\bar{\alpha}+\bar{\alpha}_0}$, where α_0 satisfies (4.1), is also a solution of (1.6), because if we act on a P-function with an Inui operator, conditions (1.3) and (1.8) are preserved, and here $P_{\bar{\alpha}}$ and $P_{\bar{\alpha}+\bar{\alpha}_0}$ can be related by a product of Inui operators (4.3ab). Putting $P_m = P_{\bar{\alpha}}$ and $P_{m+1} = P_{\bar{\alpha}+\bar{\alpha}_0}$ we determine the form of (1.6) and also that of (1.1), but in general we cannot ascribe the usual meaning to λ, ϱ, q though the coefficient of f in (1.1) is completely determined from (1.7) in each case. We cannot split the function $\varphi = \frac{q-\lambda\varrho}{p}$ in two parts, depending only on m, ξ and only on n, ξ $\left(\frac{q}{p}\right.$ and $\left.\frac{\lambda\varrho}{p}\right.$ respectively$\Big)$, if all three values α, β, γ depend on m and some of them on n too, as will happen in general if we multiply two or more Inui operators.

TABLE I

Nr	Type of Inui	$H_{i,\alpha\beta\gamma;a_0\beta_0\gamma_0}$	The form of operator	$\xi^{C^*}(1-\xi)^{D^*}\frac{d}{d\xi}\xi^{C}(1-\xi)^{D}$	Transformation ε	Type of factor
1	Ii	$H_{1\alpha\beta\gamma;\frac{1}{2}0\frac{1}{2}}$	$C_1 = -\alpha$ $C_1^* = \alpha+\frac{1}{2}$	$D_1 = -\gamma$ $D_1^* = \gamma+\frac{1}{2}$	$\frac{\mu}{\xi^{1/2}(1-\xi)^{1/2}}$	A
	Iiv	$H_{2\alpha\beta\gamma;\frac{1}{2}0\frac{1}{2}}$	$C_2 = \alpha+1-\frac{1}{h}$ $C_2^* = -\alpha-\frac{1}{2}+\frac{1}{h}$	$D_2 = \gamma$ $D_2^* = -\gamma+\frac{1}{2}$		
2	Iii	$H_{1\alpha\beta\gamma;\frac{1}{2}0-\frac{1}{2}}$	ditto	$D_1 = \gamma-1$ $D_1^* = -\gamma+\frac{3}{2}$	ditto	A
	Iiii	$H_{2\alpha\beta\gamma;\frac{1}{2}0-\frac{1}{2}}$		$D_2 = -\gamma+1$ $D_2^* = \gamma-\frac{1}{2}$		
3	IIi	$H_{1\alpha\beta\gamma;\frac{1}{2}\frac{1}{2}0}$	ditto	$D_1 = \alpha+\beta$ $D_1^* = -\alpha-\beta+1$	$\frac{\mu}{\xi(1-\xi)^{1/2}}$	A_{hyp}
	IIiv	$H_{2\alpha\beta\gamma;\frac{1}{2}\frac{1}{2}0}$		$D_2^* = -\alpha-\beta-2$ $D^2 = \alpha+\beta+3$		
4	IIii	$H_{1\alpha\beta\gamma;\frac{1}{2}-\frac{1}{2}0}$	ditto	$D_1 = \alpha-\beta-\frac{1}{h}$ $D_1^* = -\alpha+\beta+\frac{1}{h}+1$	ditto	A_{hyp}
	IIiii	$H_{2\alpha\beta\gamma;\frac{1}{2}-\frac{1}{2}0}$		$D_2 = -\alpha+\beta+\frac{1}{h}-1$ $D_2^* = \alpha-\beta-\frac{1}{h}+2$		
5	IIv	$H_{1\alpha\beta\gamma;0\frac{1}{2}\frac{1}{2}}$	$C_1 = \beta+\gamma$ $C_1^* = -\beta-\gamma+1$	$D_1 = -\gamma$ $D_1^* = \gamma+\frac{1}{2}$	$\frac{\mu}{\xi^{1/2}(1-\xi)}$	A_{hyp}
	IIviii	$H_{2\alpha\beta\gamma;0\frac{1}{2}\frac{1}{2}}$	$C_2 = -\beta-\gamma-\frac{1}{h}-1$ $C_2^* = \beta+\gamma+\frac{1}{h}+2$	$D_2 = \gamma$ $D_2^* = -\gamma+\frac{1}{2}$		

6	IIvi	$H_{1\alpha\beta\gamma;\,0-\frac{1}{h}}$	$C_1 = -\beta+\gamma-\frac{1}{h}$ $C_1^* = \beta-\gamma+\frac{1}{h}+1$	ditto	ditto	A_{hyp}
	IIvii	$H_{2\alpha\beta\gamma;\,0-\frac{1}{h}}$	$C_2 = \beta-\gamma-1$ $C_2^* = -\beta+\gamma+2$			
7	IIIi	$H_{1\alpha\beta\gamma;\,100}$	$C_1 = -\alpha$ $C_1^* = \alpha$	$D_1 = \dfrac{-\left(\alpha-\frac{1}{2h}\right)^2-\left(\beta+\frac{1}{2h}\right)^2+\left(\gamma-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}{2\alpha+1-1/h}$ $D_1^* = 1-D_1$	$\dfrac{\mu}{1-\xi}$	μ-real E_{hyp} μ-imag E
	IIIii	$H_{2\alpha\beta\gamma;\,100}$	$C_2 = \alpha-\frac{1}{h}+2$ $C_2^* = -\alpha+\frac{1}{h}-2$	$D_2 = \dfrac{\left(\alpha+2-\frac{1}{2h}\right)^2+\left(\beta+\frac{1}{2h}\right)^2-\left(\gamma-\frac{1}{2}\right)-\left(\frac{1}{2}\right)^2}{2\alpha+3-1/h}$ $D_2^* = 1-D_2$		
8	IIIiii	$H_{1\alpha\beta\gamma;\,001}$	$C_1 = \dfrac{\left(\alpha-\frac{1}{2h}\right)^2-\left(\beta+\frac{1}{2h}\right)^2+\left(\gamma-\frac{1}{2h}\right)^2+\left(\frac{1}{2h}\right)^2}{2\gamma}$ $C_1^* = 1-C_1$	$D_1 = -\gamma$ $D_1^* = \gamma$	$\dfrac{\mu}{\xi}$	ditto
	IIIiv	$H_{2\alpha\beta\gamma;\,001}$	$C_2 = \dfrac{-\left(\alpha-\frac{1}{2h}\right)^2+\left(\beta+\frac{1}{2h}\right)^2-\left(\gamma+\frac{1}{2h}+1\right)^2+\left(\frac{1}{2h}\right)^2}{2(\gamma+1)}$ $C_2^* = 1-C_2$	$D_2 = \gamma+1$ $D_2^* = -\gamma-1$		
9	IVi	$H_{1\alpha\beta\gamma;\,010}$	$C_1 = \dfrac{\left(\alpha-\frac{1}{2h}\right)^2+\left(\beta+\frac{1}{2}\right)^2-\left(\gamma-\frac{1}{2}\right)^2-\left(\frac{1}{2h}\right)^2}{2\beta+1+1/h}$ $C_1^* = 1-C_1$	$D_1 = \dfrac{-\left(\alpha-\frac{1}{2h}\right)^2+\left(\beta+\frac{1}{2h}\right)^2+\left(\gamma-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}{2\beta+1+1/h}$ $D_1^* = 1-D_1$	$\dfrac{\mu}{\xi(1-\xi)}$	ditto
	IVii	$H_{1\alpha\beta\gamma;\,010}$	$C_2 = \dfrac{-\left(\alpha-\frac{1}{2h}\right)^2-\left(\beta+\frac{3}{2}+\frac{1}{2h}\right)^2+\left(\gamma-\frac{1}{2}\right)^2+\left(\frac{1}{2h}\right)^2}{2\beta+3+1/h}$ $C_2^* = 1-C_2$	$D_2 = \dfrac{\left(\alpha-\frac{1}{2h}\right)^2-\left(\beta+2+\frac{1}{2h}\right)^2-\left(\gamma-\frac{1}{2}\right)+\left(\frac{1}{2}\right)^2}{2\beta+3+1/h}$ $D_2^* = 1-D_2$		

5. Transition to the factorization method

We ask now, what conditions must satisfy operators (4.2) in order to form, after a transformation (2.1), a factorization (2.4) of the Eq. (2.2). We obtain from (4.2)

$$A_{1\bar{a}\bar{a}_0}\left(B_{1\bar{a}\bar{a}_0}+\frac{1-1/h}{2\xi}+\frac{1}{2}\frac{\epsilon'}{\epsilon}-\epsilon\frac{d}{dv}\right)u_m = K_{1\alpha\bar{a}_0}u_{m+1} \tag{5.1a}$$

$$A_{2\bar{a}\bar{a}_0}\left(+B_{2\bar{a}\bar{a}_0}-\frac{1-1/h}{2\xi}-\frac{1}{2}\frac{\epsilon'}{\epsilon}+\epsilon\frac{d}{dv}\right)u_{m+1} = K_{2\bar{a}\bar{a}_0}u_m\,. \tag{5.1b}$$

Eqs. (5.1) have to be identical with (2.4). Then we have the following conditions

$$-A_{1\bar{a}\bar{a}_0} = C\cdot A_{2\bar{a}\bar{a}_0} \tag{5.2}$$

$$B_{1\bar{a}\bar{a}_0}-B_{2\bar{a}\bar{a}_0}+\frac{d}{d\xi}\log\epsilon = -\frac{1-1/h}{\xi} \tag{5.3}$$

$$\begin{aligned} &A_{i\bar{a}\bar{a}_0}\cdot\epsilon = \text{const} \quad (\text{for all } \bar{a},\bar{a}_0) \text{ i. e.}\\ &A_{i\bar{a}'\bar{a}_0} = A_{i\bar{a}\bar{a}_0} \quad (\text{for any } \bar{a}\neq\bar{a}')\,. \end{aligned} \tag{5.4}$$

If (5.4) is satisfied, (5.2, 3) have the same form as (4.5, 5'). (2.1) takes now the form

$$\left\{\frac{1}{\epsilon}\left(B_{1\bar{a}\bar{a}_0}+\frac{1-1/h}{2\xi}+\frac{1}{2}\frac{\epsilon'}{\epsilon}\right)\pm\frac{d}{dv}\right\}u^m_{m+1} = K_{\frac{1}{2}\bar{a}\bar{a}_0}u^{m+1}_m \tag{5.5}$$

which, again, if (5.4) is satisfied, is identical with (4.6). We see that (5.4) forms an additional condition for operators (4.2). We now ask, when is this additional condition satisfied?

All single Inui operators fulfill this condition, but if we consider a product of two operators the situation is different.

We have 18^2 possible products of two Inui operators, but only 24 lead to new relations, because a) some products of two Inui operators give another Inui operator $O_iO_j = O_k$ (if $\bar{\alpha}_{i0}+\bar{\alpha}_{j0} = \bar{\alpha}_{k0}$), b) some are identical with a product of two identical Inui operators, and then the problem can be reduced to single Inui operators (if $\alpha_{i0}+\alpha_{j0} = 2\alpha_{k0}$). For the 24 new relations, we have $|\alpha_0|+|\beta_0|+|\gamma_0| = 2$ and $\bar{\alpha}_0/2$ does not satisfy (4.1). Then we have for (4.2)

$$H_{1(\bar{a}+\bar{a}_0)\bar{a}'_0}H_{1\bar{a}\bar{a}_0}P_{\bar{a}} = K_{1(\bar{a}+\bar{a}_0)\bar{a}'_0}K_{1\bar{a}\bar{a}_0}P_{\bar{a}+\bar{a}_0+\bar{a}'_0}\,, \tag{5.6a}$$

$$H_{2(\bar{a}\bar{a}'_0)}H_{2(\bar{a}+\bar{a}'_0)}P_{\bar{a}+\bar{a}_0+\bar{a}'_0} = K_{2\bar{a}\bar{a}'_0}K_{2(\bar{a}+\bar{a}'_0)\bar{a}_0}P_{\bar{a}}\,. \tag{5.6b}$$

Using (1.6) we have

$$\begin{aligned} A_{\bar{a}(\bar{a}_0+\bar{a}'_0)} = {}& \xi^{(C^*_1+C_1)_{(\bar{a}+\bar{a}_0)\bar{a}'_0}+(C^*_1+C_1)_{\bar{a}\bar{a}_0}}(1-\xi)^{(D^*_1+D_1)_{(\bar{a}+\bar{a}_0)\bar{a}'_0}+(D^*_1+D_1)_{\bar{a}\bar{a}_0}}\\ &\times\left(\frac{C_{1(\bar{a}+\bar{a}_0)a'_0}+2C_{1a a_0}+C^*_{1a a_0}-1+1/h}{\xi}-\frac{D_{1(a+a_0)a'_0}+2D_{1a a_0}+D^*_{1a a_0}}{1-\xi}\right). \end{aligned} \tag{5.7}$$

TABLE II

Number and α_0	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(9′)	(8′)	(7′)	(6′)	(5′)	(4′)	(3′)	(2′)	(1′)
(1) $\frac{1}{2}0\frac{1}{2}$	101 *AF*	(7)	$1\frac{1}{2}\frac{1}{2}$ I	$1-\frac{1}{2}\frac{1}{2}$ II	$\frac{1}{2}\frac{1}{2}1$ III	$\frac{1}{2}-\frac{1}{2}1$ IV	$\frac{3}{2}0\frac{1}{2}$ V	$\frac{1}{2}0\frac{3}{2}$ VI	$\frac{1}{2}1\frac{1}{2}$ VII	$\frac{1}{2}-1\frac{1}{2}$ VIII	(2)	(2′)	(3)	(4)	(5)	(6)	(8)	000
(2) $\frac{1}{2}0-\frac{1}{2}$		10−1 *AF*	$1\frac{1}{2}-\frac{1}{2}$ XI	$1-\frac{1}{2}-\frac{1}{2}$ X	(3)	(4)	$\frac{3}{2}0-\frac{1}{2}$ XI	(1)	$\frac{1}{2}1-\frac{1}{2}$ XII	$\frac{1}{2}-1-\frac{1}{2}$ XIII	$\frac{1}{2}0-\frac{3}{2}$ XIV	(1′)	$\frac{1}{2}\frac{1}{2}-1$ XV	$\frac{1}{2}-\frac{1}{2}-1$ XVI	(6′)	(5′)	000	
(3) $\frac{1}{2}\frac{1}{2}0$			110 *AF*	(7)	VII	(1)	$\frac{3}{2}\frac{1}{2}0$ XVII	III	$\frac{1}{2}\frac{3}{2}0$ XVIII	(4)	XV	(4′)	XII	(2)	(9)	000		
(4) $\frac{1}{2}-\frac{1}{2}0$				1−10 *AF*	(1)	XX	$\frac{3}{2}-\frac{1}{2}0$ XIX	IV	(3)	$\frac{1}{2}-\frac{3}{2}0$ XX	XVI	(3′)	(2)	XX	000			
(5) $0\frac{1}{2}\frac{1}{2}$					011 *AF*	(8)	II	$0\frac{1}{2}\frac{3}{2}$ XXI	$0\frac{3}{2}\frac{1}{2}$ XXII	(6)	(6′)	II′	(8)	000				
(6) $0-\frac{1}{2}\frac{1}{2}$						0−11 *AF*	(14)	$0-\frac{1}{2}\frac{3}{2}$ XXIII	(5)	$0-\frac{3}{2}\frac{1}{2}$ XXIV	(5′)	I′	000					
(7) 100							200 *AF*	(11) *AF*	(33) *AF*	(44) *AF*	(22) *AF*	000						
(8) 001								002 *AF*	(55) *AF*	(66) *AF*	000							
(9) 010									020 *AF*	000								

Numbers (1)-(9) denote operators from Table I. (1′)-(9′) operators with $\bar{a}_0' = -\bar{a}_0'$ of operators (1)-(9) respectively. Products $(ik) = (ki)$ and $(ik') = (ki')'$. The "dashing" of an operator consists of interchanging the role of H_1 and H_2. *AF* — cases which can be treated by "Artificial Factorization" [2].

TABLE III

$\bar{C} = C_{1(\bar{a}+\bar{a}_0)\bar{a}_0'} + 2C_{1\bar{a}\bar{a}_0} + C^*_{1\bar{a}\bar{a}_0}$, $\bar{D} = D_{1(\bar{a}+\bar{a}_0)\bar{a}_0'} + 2D_{1\bar{a}\bar{a}_0} + D^*_{1\bar{a}\bar{a}_0}$ for operators I, II... XXIV

Nr	$\bar{C}$	$\bar{D}$
I	-2α	$\alpha+\beta-\gamma+1$
II	-2α	$\alpha-\beta-\gamma+1-1/\hbar$
III	$-\alpha+\beta+\gamma+1$	$-2\gamma-\frac{1}{2}$
IV	$-\alpha-\beta+\gamma+1-1/\hbar$	$-2\gamma-\frac{1}{2}$
V	-2α	$\dfrac{\left(\gamma+\frac{1}{2}\right)^2-\left(\alpha+\frac{1}{2}-\frac{1}{2\hbar}\right)^2+\left(\beta+\frac{1}{2\hbar}\right)^2+\left(\frac{1}{2}\right)^2+\left(\gamma-\frac{1}{2}\right)\left(2\alpha+2-\frac{1}{\hbar}\right)}{2\alpha+2-1/\hbar}$
VI	$\dfrac{\left(\alpha+\frac{1}{2}-\frac{1}{2\hbar}\right)^2-2\gamma\left(\alpha-\frac{1}{2}\right)+\left(\gamma+\frac{1}{2}-\frac{1}{2\hbar}\right)^2-\left(\beta+\frac{1}{2\hbar}\right)^2+\left(\frac{1}{2\hbar}\right)^2}{2\gamma}$	$-2\gamma-\frac{1}{2}$
VII	$-\alpha+\beta+\gamma+1$	$\alpha+\beta+\gamma-1$
VIII	$-\alpha-\beta+\gamma+1-1/\hbar$	$\alpha-\beta-\gamma+1-1/\hbar$
IX	-2α	$\alpha+\beta+\gamma$
X	-2α	$\alpha-\beta+\gamma-1/\hbar$
XI	-2α	$\dfrac{\left(\alpha+\frac{1}{2}-\frac{1}{2\hbar}\right)^2+\left(\beta+\frac{1}{2\hbar}\right)^2-(\gamma-1)^2-\left(\frac{1}{2}\right)^2+\left(\gamma-\frac{1}{2}\right)\left(2\alpha+2-\frac{1}{\hbar}\right)}{2\alpha+2-1/\hbar}$
XII	$-\alpha+\beta-\gamma+2$	$\alpha+\beta+\gamma$
XIII	$-\alpha-\beta-\gamma+2-1/\hbar$	$\alpha-\beta+\gamma-1/\hbar$
XIV	$\dfrac{\left(\alpha+\frac{1}{2}-\frac{1}{2\hbar}\right)^2-\left(\beta+\frac{1}{2\hbar}\right)^2+\left(\gamma-\frac{3}{2}+\frac{1}{2\hbar}\right)^2+\left(\frac{1}{2\hbar}\right)^2-\left(\alpha-\frac{1}{2}\right)(2\gamma-2)}{2\gamma-2}$	$2\gamma-2$

XV	$-\alpha+\beta-\gamma+2$	$2\gamma-2$
XVI	$-\alpha-\beta-\gamma+2-1/h$	$2\gamma-2$
XVII	$-2\alpha+1$	$\dfrac{-\left(\alpha+\frac{1}{2}-\frac{1}{2h}\right)^2-\left(\beta+\frac{1}{2}+\frac{1}{2h}\right)^2+\left(\gamma-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+(\alpha+\beta+1)\left(2\alpha+2-\frac{1}{h}\right)}{2\alpha+2-1/h}$
XVIII	$\dfrac{\left(\alpha+\frac{1}{2}-\frac{1}{2h}\right)^2+(\beta+1)^2-\left(\gamma-\frac{1}{2}\right)^2-\left(\frac{1}{2h}\right)^2-\left(\alpha-\frac{1}{2}\right)\left(2\beta+2+\frac{1}{2h}\right)}{2\beta+2+1/2h}$	$\dfrac{-\left(\alpha+\frac{1}{2}-\frac{1}{2h}\right)^2+\left(\beta+\frac{1}{2}+\frac{1}{2h}\right)^2+\left(\gamma-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+(\alpha+\beta+1)\left(2\beta+2+\frac{1}{2}\right)}{2\beta+2+1/h}$
XIX	$-2\alpha+1$	$\dfrac{\left(\alpha+\frac{1}{2}-\frac{1}{2h}\right)^2-\left(\beta-\frac{1}{2}+\frac{1}{2h}\right)^2+\left(\gamma-\frac{1}{2}\right)-\left(\frac{1}{2}\right)^2+\left(\alpha+\beta+1-\frac{1}{h}\right)\left(2\alpha+2-\frac{1}{h}\right)}{2\alpha+2-1/h}$
XX	$\dfrac{-\left(\alpha+\frac{1}{2}-\frac{1}{2h}\right)^2-\left(\beta-1+\frac{1}{2h}\right)^2+\left(\gamma-\frac{1}{2}\right)^2+\left(\frac{1}{2h}\right)^2-\left(\alpha-\frac{1}{2}\right)\left(2\beta-1+\frac{1}{h}\right)}{2\beta-1+1/h}$	$\dfrac{\left(\alpha+\frac{1}{2}-\frac{1}{2h}\right)^2-\left(\beta-\frac{1}{2}+\frac{1}{2h}\right)^2-\left(\gamma-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2+\left(\alpha-\beta+1-\frac{1}{h}\right)\left(2\beta-1+\frac{1}{h}\right)}{2\beta-1+1/h}$
XXI	$\dfrac{\left(\alpha-\frac{1}{2h}\right)^2-\left(\beta+\frac{1}{2}+\frac{1}{2h}\right)^2+\left(\gamma+\frac{1}{2}-\frac{1}{2h}\right)^2-\left(\frac{1}{2h}\right)^2+(\beta+\gamma+1)(2\gamma+1)}{2\gamma+1}$	-2γ
XXII	$\dfrac{\left(\alpha-\frac{1}{2h}\right)^2+(\beta+1)^2-\gamma^2-\left(\frac{1}{2h}\right)^2+(\beta+\gamma+1)\left(2\beta+2+\frac{1}{n}\right)}{2\beta+2+1/h}$	$\dfrac{-\left(\alpha-\frac{1}{2h}\right)^2+\left(\beta+\frac{1}{2}+\frac{1}{2h}\right)^2+\gamma^2-\left(\frac{1}{2}\right)^2+\left(-\gamma+\frac{1}{2}\right)\left(2\beta+2+\frac{1}{h}\right)}{2\beta+2+1/h}$
XXIII	$\dfrac{\left(\alpha-\frac{1}{2h}\right)^2-\left(\beta-\frac{1}{2}+\frac{1}{2h}\right)^2+\left(\gamma+\frac{1}{2}-\frac{1}{2h}\right)^2-\left(\frac{1}{2h}\right)^2+\left(-\beta+\gamma+1+\frac{1}{h}\right)(2\gamma+1)}{2\gamma+1}$	-2γ
XXIV	$\dfrac{-\left(\alpha-\frac{1}{2h}\right)^2-\left(\beta-1+\frac{1}{2h}\right)^2+\gamma^2+\left(\frac{1}{2h}\right)^2+\left(-\beta+\gamma+\frac{1}{h}+1\right)\left(2\beta-1+\frac{1}{h}\right)}{2\beta-1+1/h}$	$\dfrac{\left(\alpha-\frac{1}{2h}\right)^2-\left(\beta-\frac{1}{2}+\frac{1}{2h}\right)^2-\gamma^2+\left(\frac{1}{2}\right)^2+\left(-\gamma+\frac{1}{2}\right)\left(2\beta-1-\frac{1}{h}\right)}{2\beta-1+1/h}$

If (5.6) gives a factorization, $A_{\bar{a}(\bar{a}_0+\bar{a}'_0)}$ cannot depend on m otherwise than multiplicatively: $A_{\bar{a}(\bar{a}_0+\bar{a}'_0)} = f(m)A_{\bar{a}_0+\bar{a}'_0}$, because then we could not speak of a transformation (2.1) valid for both u_m and u_{m+1}. Calculating $\bar{C} = C_{1(\bar{a}+\bar{a}_0)\bar{a}'_0} + 2C_{1a\bar{a}_0} + C^*_{1\bar{a}\bar{a}_0} - 1 + 1/h$ and $\bar{D} = D_{1(\bar{a}+\bar{a}_0)\bar{a}'_0} + 2D_{1\bar{a}\bar{a}_0} + D^*_{1a\bar{a}_0}$ for the 24 new operators, we obtain.

We see that for all 24 operators A depends on m otherwise than it was allowed. These operators do not give factorizations, though they lead to solutions of some problems by the polynomial method.

In Table I we showed that single Inui operators lead to all types of factorizations for the finite interval (types A, E ordinary and hyperbolic). All these problems can also be treated by the simplified polynomial method. Products of two Inui operators lead in general to problems that can be treated by the polynomial method, but cannot be factorized. In these cases we must treat the coefficient of f in (1.1) as a function of m, n, ξ that cannot be split in two parts: q/p, independent of n, and $\varrho\lambda/p$ depending only on n and ξ.

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A Model for Investigation of the Optical Properties of Noble Metals

by

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The results of numerous experiments on the optical properties of metals (cf. [8], [9], [14]) may justify suggesting a model which would allow to predict the dispersion curve for these metals at least in a limited range of frequencies.

In the region between wave-lengths 10μ to 1μ the Drude theory predicts the optical constants with some success. In this region the dielectric coefficient agrees reasonably well with values deduced from experimental data, but the absorption coefficient is always too small. For wave-lengths below 1μ, particularly down to 0.1μ, photoelectric absorption sets in, and it is in this region that the optical constants should depend most strongly on the band structure of the metal. The highest occupied valence band and the next unoccupied bands should play the predominant role here.

The theory of photoelectric interband absorption as related to the band structure of a solid was put forward by Kronig [12], Wilson [16] and others. It was later developed by Sergeiev and Tchernikovsky [15], by Fan [6], and by Butcher [4] and applied with success to the alkali metals. The necessary assumptions about the band structure were made on the basis of the nearly free electron approximation. It is the intention of the author to propose a corresponding theory for the optical constants of noble metals. Again the nearly free electron approximation was helpful, but emphasis has been laid on the bending of energy surfaces near the boundary of the Brillouin zone, since in noble metals, because of their f. c. c. structure, the Fermi surface may well touch the boundary or approach it closely. As is well known, the bending of energy surfaces near the Brillouin zone boundary results in a pronounced modification of the dielectric coefficient and the absorption coefficient [16] and also of other properties depending on the density of states such as thermo-

electric power [10] and specific heat [11] and magnetic susceptibility [3]. The simple model proposed here was intended for the purpose of studying the modification of the optical properties in a more quantitative way.

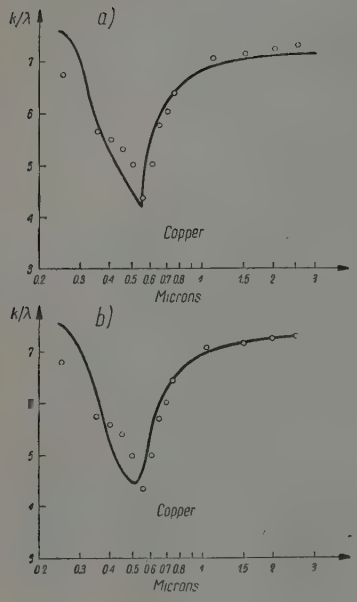

Fig. 1

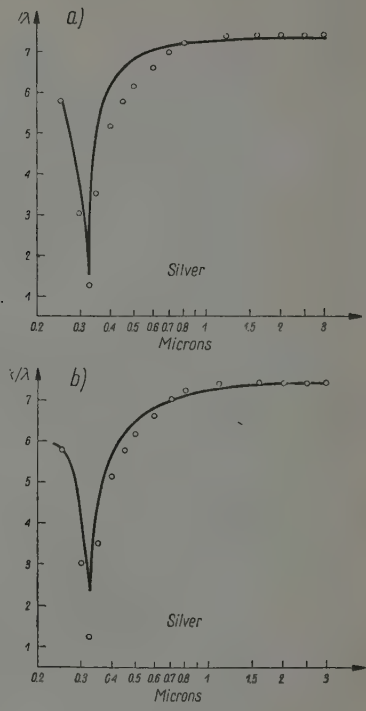

Fig. 2

Fig. 1. Graph showing k/λ values for copper as a function of wave-length. The circles denote Schulz's experimental points. The curve is computed with the Fermi surface having a distance from the L point — $(a)\, h = 0$, $(b)\, h = 0.05 k_0$.

Fig. 2. Graph showing k/λ values for silver as a function of wave-length λ. The circles denote Schulz's experimental points. The curve is computed with the Fermi surface having a distance from the L point — $(a)\, h = 0$, $(b)\, h = 0.05 k_0$.

The need for such a study emerges in connection with several recent papers (cf. [5], [13], [17], [18]).

A two-band model has been considered. The energy surfaces for the two energy bands, lower (−) and upper (+) have been taken in the form

$$E_{\pm}(k_x, k_y, k_z) = \frac{\hbar^2}{2m}[k_x^2 + k_y^2 + (k_0 - k_z)^2 + k_0^2 \pm 2k_0\sqrt{(k_0 - k_z)^2 + V^2}]\,.$$

Here k_x, k_y, k_z are the components of the wave vector with the z-axis taken along the ΓL line which is the line from the origin Γ to the centre L of the hexagonal face of the Brillouin zone. The constant k_0 is the distance ΓL in the reciprocal space and V is proportional to the energy gap at the point L.

The surface containing the occupied states has been taken in the form $E_- = E_F$, where the constant E_F is the Fermi energy. The matrix elements for the momentum operator have been assumed constant. With these assumptions the necessary summations, indicated, for example, by Wilson [16], can be carried out analytically. The resulting dielectric coefficient and absorption coefficient have been added to the respective Drude coefficients, and the extinction coefficient $k(\lambda)$ has been calculated. The curves for copper and silver were compared with the recent experimental data of Schulz [14] (see Fig. 1 and Fig. 2), where the wave-lengths are represented in microns by the abscissae and the quantity $k(\lambda)/\lambda$ by the ordinates.

The calculated dielectric and absorption coefficients are finite for Fermi surfaces inside the Brillouin zone but they exhibit infinities in

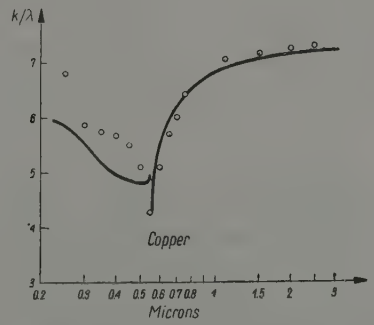

Fig. 3. Graph showing k/λ values for copper as a function of wave-length λ. The circles denote Schulz's experimental points. The curve is computed with the Fermi surface having an area of contact A with the hexagonal face of the Brillouin zone boundary — $A = 0.0036\pi k_0^2$.

the present model if the Fermi surface has a finite area of contact with the Brillouin zone boundary. An example of such a situation calculated with an area of contact $A = 0.0036\ \pi k_0^2$ is presented in Fig. 3.

The dielectric and absorption coefficients can be made finite by the usual assumption that the transverse effective masses in the lower and upper bands are not exactly equal. The resulting expressions have been derived but are very lengthy. They may be of importance if the optical absorption in noble metals is related to some excitation of the filled d band [7].

When the Fermi surface is moved inwards away from the zone boundary, the peak in the dielectric coefficient curve becomes smaller in absolute value so that it is difficult to account for the whole dip observed in the curve of the extinction coefficient. Also, to produce the dip at the wave-length given by experiment it is necessary to assume a correspondingly smaller value for V, since the frequency, at which the extinction coefficient curve is a minimum, is determined by the energy gap between the lower and upper energy band at the highest filled state. This frequency is given exactly by the energy gap at the point L, when the Fermi surface just touches the Brillouin zone boundary, and is larger than this gap, when the Fermi surface is inside the Brillouin zone. Only vertical optical transitions are considered here.

The width of the minimum in the dispersion curve is determined mainly by the spread of the energy bands involved. For copper, where the ratio of the energy gap (taken 2.25 ev) to the free electron energy at the point L (8.67 ev) is small, the dip in the computed curve is broad, too broad, in fact. For silver, where this ratio has been taken as 3.81:6.76, it is too narrow.

If, in the present model, the constant matrix element for the momentum operator is replaced by some element which varies in k-space, the width of the dip in the k/λ curve changes. If the absolute value of the matrix element diminishes away from the Brillouin zone boundary towards the centre of the zone, the resulting minimum in the k/λ curve becomes narrower.

Starting from the analysis of the experimentally determined optical constants it seems rather difficult to make a definite statement about the touching of the Brillouin zone boundary. One conclusion which can certainly be drawn from the present investigation is, that it is exceedingly important to take the energy gap and the consequent bending of energy surfaces into account. The dip in the k/λ curve can be simply related to the increasing density of states near the Brillouin zone boundary. The situation is similar to that in the superconductor where the energy gap exists in the micro-wave region of frequencies (in connection with this problem see [1] and [2]). The effectively observed extinction coefficient depends very sensitively on the dielectric coefficient for wave-lengths in the vicinity of the threshold for interband absorption.

I thank Professor H. Jones for suggesting this investigation and for many valuable discussions, and also for his kind hospitality extended to me at the Imperial College.

The author wrote this note while on leave of absence from the Institute of Theoretical Physics of Warsaw University.

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Über die Bewegungsgleichungen in linearen Feldtheorien

von

C. JANKIEWICZ

Vorgelegt von L. INFELD am 26. Januar 1959

1. In der Gravitationstheorie folgen die Feldgleichungen aus der Forderung der Stationärität einer entsprechenden Wirkungsfunktion gegenüber der Variation des metrischen Tensors $g_{\alpha\beta}(x^\gamma)$. (Die mit kleinen griechischen Buchstaben bezeichneten Indizes durchlaufen die Werte 1, 2, 3, 4). Aus diesen Gleichungen kann man, bei manchen zusätzlichen Annahmen, die die Wahl des Bezugssystems betreffen, das Feld und die Bewegung seiner Quellen bestimmen. In der Methode von Infeld [1] ist das Bezugssystem durch die Anwendung eines speziellen, Näherungsverfahrens bestimmt, in der Fockschen [2] Methode dagegen, mittels der Harmonitätsbedingungen.

In der allgemeinen Relativitätstheorie charakterisiert der Tensor $g_{\alpha\beta}$ nicht nur das Gravitationsfeld, sondern auch das Bezugssystem (Koordinatensystem). Wir bestimmen also das Feld und die Bewegung seiner Quellen aus den das Bezugssystem festlegenden Bedingungen und den Gleichungen, die die Forderung der Stationärität einer entsprechenden Wirkungsfunktion gegenüber der Variation der das Gravitationsfeld und das Bezugssystem beschreibenden Funktionen liefern. In der vorliegenden Note wird gezeigt, dass das elektromagnetische Feld sowie die Bewegungen seiner Quellen aus den das Bezugssystem festlegenden Bedingungen und aus den Gleichungen, zu denen die Forderung der Stationärität einer entsprechenden Wirkungsfunktion gegenüber der Variation der Potentiale des elektromagnetischen Feldes und der das Bezugssystem beschreibenden Funktionen führt, bestimmt werden können. Auf ähnliche Weise kann das Feld und die Feldquellenbewegung in jeder linearen Feldtheorie bestimmt werden. So haben wir das Problem der Ableitung der Bewegungsgleichungen aus den Feldgleichungen in linearen Feldtheorien gelöst. Wir haben es anders aufgestellt, als es Infeld in seiner Arbeit [3] getan hat, deren Grundgedanke auf dem Grenzübergang mit den Gravitationsfeldgleichungen und den Gleichungen des Zusatzfeldes zur speziellen Relativitätstheorie beruht.

2. Das elektromagnetische Feld beschreiben wir durch den kovarianten Vektor $A_\alpha(x^\nu)$, das Bezugssystem — durch vier skalare Funktionen $\overset{\alpha}{\psi}(x^\nu)$, die wir Trägheitsfeldpotentiale nennen werden. Im Inertialbezugssystem haben diese Funktionen die Gestalt

$$(1) \qquad \overset{\alpha}{\psi} = \overset{\alpha}{C}_\mu x^\mu + \overset{\alpha}{C}, \qquad \sum_{\alpha=1}^{4} \overset{\alpha}{C}_\mu \overset{\alpha}{C}_\nu = \delta_{\mu\nu} = \begin{matrix} 1 & \text{für} & \mu = \nu \\ 0 & \text{für} & \mu \neq \nu \end{matrix},$$

wo $\overset{\alpha}{C}_\mu$ und $\overset{\alpha}{C}$ Konstante sind. Aus ihnen bauen wir den symmetrischen metrischen Tensor

$$(2) \qquad \gamma_{\mu\nu} = \sum_{\alpha,\beta=1}^{4} \varepsilon_{\alpha\beta} \frac{\partial \overset{\alpha}{\psi}}{\partial x^\mu} \frac{\partial \overset{\beta}{\psi}}{\partial x^\nu}$$

auf, wo $\varepsilon_{\alpha\beta}$ den metrischen Tensor des Minkowski-Raumes bezeichnet.

Elektrisch geladene Körper fassen wir als Feldsingularitäten auf, die wir mittels der von Infeld und Plebański [4] modifizierten Diracschen δ-Funktion $\delta(x-\xi_i) = \hat{\delta}(\bar{x}-\bar{\xi}_i)\delta(t-\tau_i)$ beschreiben werden. Die Linien $\xi_i^\alpha(\lambda_i)$, $i = 1, 2, \ldots, N$, wo λ_i beliebige Parameter sind, stellen die Bewegungen der einzelnen Körper dar; $\dot{\xi}_i^\alpha = d\xi_i^\alpha/d\lambda_i$ sind die Körpergeschwindigkeiten.

Die Wirkungsfunktion nehmen wir für das Feld und seine Singularitäten in der Form

$$(3) \qquad S = \int \left[\int \sum_i \delta(x-\xi_i) \left(m_i \sqrt{\gamma_{\alpha\beta} \dot{\xi}_i^\alpha \dot{\xi}_i^\beta} - \frac{e_i}{c} A_\alpha \dot{\xi}_i^\alpha \right) d\lambda_i + \frac{1}{16\pi} H_{\alpha\beta} H^{\alpha\beta} \sqrt{-\gamma} \right] (dx)$$

an, wo

$$H_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha, \qquad H^{\mu\nu} = \gamma^{\mu\alpha}\gamma^{\nu\beta} H_{\alpha\beta}, \qquad \gamma = \det(\gamma_{\alpha\beta})$$

und m_i, e_i und c physikalische Konstanten bedeuten (Körpermassen, Körperladungen und Lichtgeschwindigkeit). Das Symbol ∇_α bezeichnet die mit dem metrischen Tensor $\gamma_{\alpha\beta}$ kovariante Ableitung. (Über sich wiederholende mit kleinen griechischen Buchstaben bezeichnete Indizes sollen von 1 bis 4 summiert werden).

3. Die Variation der Wirkungsfunktion (3) nach der Funktion ξ_i^α hat die Gestalt

$$(4) \qquad \delta[\xi_i^\alpha] S = \int \left[m_i \left(\frac{d\eta_i^\mu}{ds_i} + \tilde{\Gamma}^\mu_{i\alpha\beta} \eta_i^\alpha \eta_i^\beta \right) - \frac{e_i}{c} \tilde{H}^\mu_{i\alpha} \eta_i^\alpha \right] \tilde{\gamma}_{i\mu\nu} \delta\xi_i^\nu ds_i,$$

wo

$$\eta_i^\alpha = d\xi_i^\alpha/ds_i, \qquad ds_i = \sqrt{\tilde{\gamma}_{i\alpha\beta} \dot{\xi}_i^\alpha \dot{\xi}_i^\beta}\, d\lambda_i, \qquad \tilde{\varphi}_i = \int \varphi \hat{\delta}(\bar{x}-\bar{\xi}_i)(dx).$$

Aus der Formel (4) erhalten wir die Bewegungsgleichung

$$(5) \qquad m_i \left(\frac{d\eta_i^\mu}{ds_i} + \tilde{\Gamma}^\mu_{i\alpha\beta} \eta_i^\alpha \eta_i^\beta \right) + \frac{e_i}{c} \tilde{H}^\mu_{i\alpha} \eta_i^\alpha = 0.$$

Die Variation der Wirkungsfunktion (3) nach den Funktionen A_α und $\overset{a}{\psi}$ gibt

$$(6)\qquad \delta[A_\alpha, \overset{a}{\psi}]S = \int \left\{ \left[\sqrt{-\gamma}\, \nabla_\nu H^{\nu\mu} + \frac{4\pi}{c} \int \sum_i e_i \delta(x-\xi_i) \dot{\eta}_i^\mu ds_i \right] \delta A_\mu + \right.$$

$$\left. + \left[\nabla_i \left(\sqrt{-\gamma}\, U^{\mu\nu} + \int \sum_i m_i \delta(x-\xi_i) \dot{\eta}_i^\mu \dot{\eta}_i^\nu ds_i \right) \right] \gamma_{\mu\varrho} \delta\psi^\varrho \right\} (dx) ,$$

wo

$$\delta\psi^\varrho = \sum_{\alpha,\beta=1}^{4} \varepsilon_{\alpha\beta} \delta\overset{a}{\psi} \frac{\partial \overset{\beta}{\psi}}{\partial x^\sigma} \gamma^{\sigma\varrho}$$

und

$$(7)\qquad U^{\mu\nu} = -\frac{1}{4\pi} \gamma_{\alpha\beta} H^{\mu\alpha} H^{\nu\beta} + \frac{1}{16\pi} H_{\alpha\beta} H^{\alpha\beta} \gamma^{\mu\nu}$$

der Impuls – Energie – Tensor des elektromagnetischen Feldes ist. Aus (6) erhalten wir die Gleichungen für das verallgemeinerte elektromagnetische Feld, d. h. für das elektromagnetische und das Trägheitsfeld

$$(8)\qquad \nabla_\nu H^{\mu\nu} + \frac{4\pi}{c\sqrt{-\gamma}} \int \sum_i e_i \delta(x-\xi_i) \dot{\eta}_i^\mu ds_i = 0 ,$$

$$(9)\qquad \nabla_\nu \left(\sqrt{-\gamma}\, U^{\mu\nu} + \int \sum_i m_i \delta(x-\xi_i) \dot{\eta}_i^\mu \dot{\eta}_i^\nu ds_i \right) = 0 .$$

Bei Beachtung der Identitäten

$$(10)\qquad \nabla_\mu U^{\mu\nu} - \frac{1}{4\pi} H_{\nu\mu} \nabla_\alpha H^{\mu\alpha} = 0$$

erhält man die Bewegungsgleichungen (5), wenn man die Gleichungen (8) und (9) über sich nicht überlappende Bereiche um die Singularitäten integriert.

4. Im oben betrachteten Fall des elektromagnetischen Feldes ist es leicht das bevorzugte Bezugssystem zu finden.

Es ist eben das Inertialbezugssystem, in welchem die Trägheitsfeldpotentiale durch die Formeln (1) gegeben sind. In einem solchen Bezugssystem haben die Gleichungen des verallgemeinerten elektromagnetischen Feldes die Gestalt

$$(11)\qquad \partial_\nu H^{\mu\nu} + \frac{4\pi}{c} \int \sum_i e_i \delta(x-\xi_i) \dot{\eta}_i^\mu d\varepsilon_i = 0 ,$$

$$(12)\qquad \partial_\nu \left(U^{\mu\nu} + \int \sum_i m_i \delta(x-\xi_i) \dot{\eta}_i^\mu \dot{\eta}_i^\nu ds_i \right) = 0 ,$$

wo $ds_i^2 = \varepsilon_{\alpha\beta} d\xi_i^\alpha d\xi_i^\beta$. Bei entsprechenden Randbedingungen haben sie eindeutige Lösungen. Wir sehen also, dass die von Infeld und Wallace [5] gegebene Problemstellung der Ableitung der Bewegungsgleichungen aus den Gleichungen des elektromagnetischen Feldes äquivalent der Ableitung der Bewegungsgleichungen aus den Gleichungen des verrallgemeinerten elektromagnetischen Feldes im Inertialbezugssystem ist.

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EXPERIMENTAL PHYSICS

On a Possible Explanation of Strong Photovoltaic Effects in PbS Layers

by
T. FIGIELSKI

Presented by A. SOŁTAN on December 29, 1958

The earliest investigations of photovoltaic effects in microcrystalline layers are due to J. Starkiewicz and co-workers [1] and date back to 1946. These authors found photoelectromotive forces EMF amounting to several volts in PbS layers that had been formed electrically.

The effect was explained by L. Sosnowski by asymmetry of the intercrystalline barriers, arising from the forming current.

Recently, R. J. Berłaga and others [2] obtained photovoltaic PbS layers exhibiting an EMF of up to 3V without previous electrical formation of the layers. The latter had been obtained by evaporation of PbS onto glass and subsequent heating to 500-600°C in an atmosphere of various gases. Thermal processing in an atmosphere of air raised the photoelectromotive force to a value of several volts.

These layers exhibited a pronounced dependence of the magnitude and sign of the EMF on the direction of the incident light beam, the direction of inversion of the EMF corresponding to that in which the layer had been deposited.

A similar isotropic strong photovoltaic effect had been observed in Tl_2S layers by A. Wolska [3]. Investigation with the electron microscope [4] proved these layers to consist of small needle-shaped crystals oriented in the direction in which the vapour had been deposited. The authors of the paper in question suggest that EMF is generated in these layers in connection with the appearance of an external photoeffect on the illuminated surfaces of the crystals. However, such a process should hardly seem probable for a number of reasons. It is the present author's opinion that the phenomenon consists in a typical photovoltaic effect on the intercrystalline boundaries, such as should occur when the latter are non-symmetrically illuminated.

A photovoltaic effect of this latter type was previously investigated and discussed by the present author in the case of germanium [5]. As-

sume that a symmetrical potential barrier of height Φ is formed on the grain boundaries (Fig. 1a). The photoelectromotive force generated on such a barrier at unilateral illumination of the boundary may be expressed by the following relationship valid for weak illumination:

$$V = \frac{kT}{e} \frac{\Delta n_{\min}}{n_{\min}} S ,$$

where $n_{\min}$ denotes the concentration of the minority carriers, $\Delta n_{\min}$ the rise in concentration of the minority carriers caused by illumination, and $S = \left(1 + \frac{I_{\text{maj}}}{I_{\min}}\right)^{-1}$ – a quantity accounting for the degree of separation of the pairs of current carriers on the barrier. Herein, $I_{\text{maj}}/I_{\min}$ is the ratio of majority and minority currents crossing the barrier at low bias of the latter.

In general, the minority current across the barrier implies processes of generation-recombination and is independent of the height of the barrier. The majority current may be treated similarly as in the theory of Schottky's barrier layer or in diode theory. In every case, it is proportional to the concentration of the majority carriers at the top of the barrier:

$$I_{\text{maj}} \sim \exp(-e\Phi/kT).$$

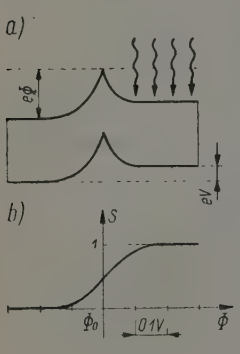

Fig. 1

As a result of this, the coefficient S of separation of pairs on the barrier plotted as a function of Φ yields a curve of the type shown in Fig. 1b.

S approaches unity as the height of the barrier exceeds a certain minimum value Φ_0. At $S \simeq 1$, the EMF generated on a single boundary, with intense illumination, may be of the order kT/e. Within the layer, asymmetry of illumination of the intercrystalline boundaries is provided by the small protruding needle-shaped crystals (of a length of several μ), which create regions or shadow. In reality, the asymmetry, and hence the EMF, is lowered owing to the fact that the effective diffusion length of the carriers generated may be comparable to the dimensions of those regions. It may be assumed that, in a layer, the number of asymmetrically illuminated barriers per 1 mm is approximately 1000. The EMF generated on each may be of the order of 1 m V, which is quite sufficient for explaining the values of the photoeffect observed in the layers. The considerable rise in the photoelectromotive force taking place during thermal processing should be related to the formation and rise in height of barriers

at the intercrystalline surfaces. Such barriers may arise, e. g. as a result of the adsorption of oxygen at the crystal surfaces.

The author wishes to express his indebtedness to Professor L. Sosnowski for his critical remarks on the interpretation proposed.

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М. СУФФЧИНСКИЙ, МОДЕЛЬ ПОСТРОЕННАЯ ДЛЯ ИССЛЕДОВАНИЙ ОПТИЧЕСКИХ СВОЙСТВ БЛАГОРОДНЫХ МЕТАЛЛОВ стр. 169—173

Оптические константы металлов были, до настоящего времени, исчисляемы при приблизительном предположении, что электроны почти свободны. Оказывается, однако, что оптические константы зависят в сильной степени от кривизны энергетических поверхностей вблизи границ зоны Бриллуина. В работе рассматривается метод исчисления оптических констант, полагая такой вид функции энергии, который учитывает влияние границ зоны Бриллуина.

Исчислены: диэлектрическая постоянная и коэффициент абсорбции, а затем коэффициент экстинкции. Пробег коэффициентов экстинкции для меди и серебра, в засисимости от длины волны, показан на графиках.

В заключение дискутируется, каким образом изменяется коэффициент экстинкции, когда поверхность Ферми принимает разные положения по отношению к границам зоны Бриллуина.

Ч. ЯНКЕВИЧ, ОБ УРАВНЕНИЯХ ДВИЖЕНИЯ В ЛИНЕЙНЫХ ТЕОРИЯХ ПОЛЯ . . . . . стр. 175—178

Показано, что уравнения движения частиц, обладающих электрическими зарядами и рассматриваемых как особенности поля, следуют из уравнений, выводимых из вариационного принципа при произвольных вариациях электромагнитных потенциалов и функций, характеризующих систему отсчета.

Т. ФИГЕЛЬСКИЙ, НЕКОТОРОЕ ПОЯСНЕНИЕ СИЛЬНЫХ ЭФФЕКТОВ ФОТОВОЛЬТАИЧЕСКИХ В СЛОЯХ PbS . . . . . стр. 179—181

В слоях PbS, напыленных паром в вакууме и подверженных термической обработке, наблюдается фотоэлектродвижущаяся сила, которой величина зависит от направления падающего светового луча и может доходить до нескольких вольт.

Автор пытается выяснить такого рода фото-эдс при помощи фотовольтаических эффектов, выступающих на границах зерен при несимметрическом их освещении. В случае достаточно высоких межкристаллических барьеров, обеспечивающих эффективное разделение пар носителей тока, эти эффекты будут заметны сильнее.

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

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Ф. А. САС, ЯВНОЕ ОПРЕДЕЛЕНИЕ НЕКОТОРЫХ КЛАССОВ АССОЦИАТИВНЫХ КОЛЕЦ . . . . . стр. 107–110

В работе дается (без доказательства) определение всех классов колец R, выполняющих одно из нижеприведенных условий:

Условие E_1. Все отличающиеся от 0 и R подкольца кольца R, обладающие конечной системой генераторов — изоморфны.

Условие E_2. Все отличающиеся от R подкольца кольца R суть вида $R \cdot r$ (где $R \in r$).

Условие E_3. Все отличающиеся от R подкольца кольца R суть левосторонние основные идеалы $Rr + Ir$ (где $r \in R$) кольца R.

Условие E_4. Все левосторонние идеалы кольца R суть вида $L = Re + L_1$, где L_1 обладает исключительно непотентными элементами, и где $e^2 = e$.

В. СЛОВИКОВСКИЙ, ПСЕВДОМЕТРИЧЕСКИЕ ПОЛУСТРУКТУРЫ . . . . . стр. 111–117

В работе вводится понятие так наз. псевдометрических полуструктур. Псевдометрической полуструктурой называем множество с определенной идемпотентной ассоциативной операцией xy и псевдометрикой ϱ, удовлетворяющей условию

$$\varrho(ac, bc) \leqslant \varrho(a, b).$$

Доказан ряд теорем, в особенности теорем с псевдометрическими полуструктурами, обладающих аддитивной псевдометрикой, т. е. таких, что $ab = b$, $bc = c$ имплицирует $\varrho(a, c) = \varrho(a, b) + \varrho(b, c)$.

Эти теоремы имеют особое значение для теорий меры и интеграла и дают общий подход к некоторым разделам этих теорем.

В. СЛОВИКОВСКИЙ, О РЕГУЛЯРНЫХ ПОГРУЖЕНИЯХ ГРУПП С ДИФФЕРЕНЦИАЛЬНЫМИ ОПЕРАТОРАМИ . . . . . стр. 119–124

Работа является продолжением и обобщением исследований, касающихся алгебраического аспекта теории обобщенных функций. Вводится понятие группы

с дифференциальными операторами, т. е. группы X с семейством операторов $\mathcal{R}$, преобразовывающих подгруппы X в X.

При предположении, что $\mathcal{R}$ — кольцо, собственно говоря, не имеющее ничего общего с сложением и суммированием операторов в группе, исследуются так наз. регулярные погружения X в $\mathcal{R}$ модули.

Гомоморфизм h группы X в $\mathcal{R}$—модуль Y называем регулярным погружением, если для $A \in \mathcal{R}$ имеется равенство $h(Ax) = A(hx)$, при предположении, что левая сторона равенства существует.

В. СЛОВИКОВСКИЙ, КАТЕГОРИИ И РЕПРЕЗЕНТАЦИИ ГРУПП С ДИФФЕРЕНЦИАЛЬНЫМИ ОПЕРАТОРАМИ . . . . . стр. 125–130

В работе рассматриваются категории в смысле Мак-Лейна регулярных погружений групп с дифференциальными операторами.

Понятие групп с дифференциальными операторами, а также регулярных погружений таких групп — было дано в работе [3]. Категории регулярных погружений позволяют исследовать, с алгебраической точки зрения, топологические группы с дифференциальными операторами. Кроме того, в работе вводится понятие репрезентации группы с дифференциальными операторами и в абстрактной форме доказывается теорема о существовании дистрибуции.

М. КШИЖАНСКИЙ, НЕКОТОРЫЕ НЕРАВЕНСТВА, КАСАЮЩИЕСЯ РЕШЕНИЙ ПАРАБОЛИЧЕСКОГО, ЛИНЕЙНОГО, НОРМАЛЬНОГО УРАВНЕНИЯ . . . . . стр. 131–135

Автор предполагает, что в некоторой области D, неограниченной и заключенной между плоскостями $t = 0$ и $t = T\,(T>0)$, коэффициенты $a_{ij}(X,t)$ параболического линейного уравнения

$$(1)\qquad F[u] = \sum_{i,j=1}^{m} a_{ij}(X,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{m} b_j(X,t)\frac{\partial u}{\partial x_j} + c(X,t)u - \frac{\partial u}{\partial t} = f(X,t),$$

ограничены, а коэффициенты $c(X,t)$ и $b_j(X,t)$ удовлетворяют соответственно неравенствам

$$(2)\qquad c(X,t) \leqslant \alpha \sum_{j=1}^{m} x_j^2 + \beta, \quad (X,t) \in D,$$

$$|b_j(X,t)| \leqslant \bar{\alpha} \sum_{j=1}^{m} |x_j| + \bar{\beta}, \quad (j = 1, 2, \ldots, m).$$

Через $\sum$ обозначаем сумму части границы $F(D)$ области D, лежащей на характеристике $t = 0$ и части ориентированной во времени (боковой поверхности).

Автор доказывает, что если функция $u(X,t)$ принадлежит к классу C^2 и удовлетворяет уравнению (1) в области D, кроме того, если она непрерывна

в множестве $D+\sum$ и удовлетворяет в этом множестве неравенству вида $|u(X,0)| \leqslant M \exp K \sum_{j=1}^{m} x_j^2$ (где M и K постоянные, зависящие в общем от функции $u(X,t)$) и если, наконец, выполняются неравенства $f(X,t) \leqslant 0$ для $X \in D$ и $u(X,t) \geqslant 0$ для $X \in \sum$, то имеется неравенство $u(X,t) \geqslant 0$ для $X \in D$.

Затем, автор представляет некоторые следствия этой теоремы.

Ч. ОЛЕХ, ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ СИСТЕМЫ ДВУХ ДИФФЕРЕНЦИАЛЬНЫХ ОБЫКНОВЕННЫХ УРАВНЕНИЙ . . . . . стр. 137–140

В работе автор рассматривает систему двух дифференциальных уравнений с правыми периодическими сторонами с периодом 1 по отношению к независимой переменной t и приводит некоторое условие, достаточное для того, чтобы существование ограниченного решения повлекло за собой существование периодического решения с периодом 1. Этот результат является обобщением (полученным другим методом) аналогичного результата З. Опяля [1], касающегося дифференциального уравнения второго порядка.

С. БАЛЬЦЕЖИК, О ФАКТОРГРУППАХ НЕКОТОРЫХ ПОДГРУПП ПОЛНОЙ ПРЯМОЙ СУММЫ БЕСКОНЕЧНЫХ ЦИКЛИЧЕСКИХ ГРУПП . . . . . стр. 141–142

Пусть абелева группа S^* (S) является полной (дискретной) прямой суммой счетного множества бесконечных циклических групп $S^* = C_\infty^{\aleph_0^*}$ ($S = C_\infty^{\aleph_0}$).

В работе дается нижеследующая теорема, ее обобщение и прямое следствие.

Теорема. *Факторгруппа S^*/S алгебраически компактная группа и имеет вид $R^{2^{\aleph_0}} + \sum_p^* I_p^{\aleph_0^*}$ (R — группа рациональных чисел, I_p — группа целых p-адических чисел).*

Э. СОНСЯДА, ДОКАЗАТЕЛЬСТВО, ЧТО ВСЯКАЯ СЧЕТНАЯ И РЕДУЦИРОВАННАЯ АБЕЛЕВА ГРУППА БЕЗ КРУЧЕНИЯ ЯВЛЯЕТСЯ УЗКОЙ . . . . . стр. 143–144

Абелева группа без кручения H называется у з к о й, если всякий гомоморфизм, отображающий полную прямую сумму счетного множества бесконечных циклических групп $\{e_\nu\}$ ($\nu = 1, 2, \ldots$) в группу H, отображает почти все компоненты $\{e_\nu\}$ в нуль группы H.

В работе дается доказательство теоремы, упомянутой в заглавии настоящей заметки.

Э. СОНСЯДА, ОБ ИЗОМОРФИЗМЕ РАЗЛОЖЕНИЙ АБЕЛЕВЫХ ГРУПП БЕЗ КРУЧЕНИЯ В ПОЛНУЮ ПРЯМУЮ СУММУ ГРУПП РАНГА 1 . . . . . стр. 145–149

Каноническим разложением абелевой группы без кручения G называется разложение $G = \sum_{\tau \in \Omega}^* G^{(\tau)}$ в полную прямую сумму групп $G^{(\tau)}$ ($\tau \in \Omega$), где

$G^{(\tau)}$ — полная прямая сумма групп ранга 1 фиксированного типа τ, а Ω множество всех типов.

Доказываются следующие теоремы:

Теорема 1. *Всякие два канонические разложения абелевой группы без кручения являются изоморфными.*

Теорема 2. *Если мощность абелевой группы без кручения G меньше мощности первого кардинального числа ненулевой меры, то всякие два разложения группы G в полную прямую сумму групп ранга 1 являются изоморфными.*

К. МОРЭН, СФЕРИЧЕСКИЕ ОБОБЩЕННЫЕ ФУНКЦИИ НА ОДНОРОДНЫХ ПРОСТРАНСТВАХ. ОБОБЩЕНИЕ ТЕОРЕМЫ БОХНЭРА-ВЕЙЛЯ-РАЙКОВА . . . . . стр. 151—155

В работе даются два применения приведенной в [6] теоремы о разложении положительно определенных ядер по обобщенным собственным функциям, а именно:

1. Инвариантное положительно определенное ядро $\theta(x, y)$ порядка $m = 0, 1, \ldots, \infty$ на однородном пространстве (Ω_n, G), где G является сепарабельной, связной группой Ли, можно разложить на (обобщенные) сферические ядра порядка $\leqslant m + \left[\frac{n}{2}\right] + 1$.

2. Если многообразие Ω_n является пространством группы Ли G, тогда получается теорема о разложении следа (в смысле Годеманта) по характерам фактор-представления. Характеры эти являются собственными обобщенными функциями $\left(\text{порядка} \leqslant m + \left[\frac{n}{2}\right] + 1\right)$ операторов Лапласа на G.

С. КРУЛИКОВСКАЯ, СООТНОШЕНИЯ МЕЖДУ МЕТОДОМ ПОЛИНОМОВ И МЕТОДОМ ФАКТОРИЗАЦИИ . . . . . стр. 157—168

Целью работы является показать соотношения, имеющиеся между двумя методами решения задач собственных значений, а именно — между методом полиномов, данным в упрощенном виде Рубиновичем [1] и методом факторизации в виде данном Инфельдом [2] для случая конечного интервала. Опираясь на рекуррентные формулы для функции P, данные Инуи'ем [3], автор доказывает, что для отдельных операторов Инуи, связывающих P_m и P_{m+1}, множества этих функций дают собственные функции задач, решаемых как методом полиномов, так и методом факторизации.

Однако, уравнения, собственные функции которых связаны между собой суперпозицией двух операторов Инуи, не во всех случаях поддаются факторизации, хотя они и разрешимы методом факторизации.

Операторы Инуи приводят ко всем видам факторизации для конечного интервала. Отсюда следует, что каждая факторизация приводима к виду, разрешимому методом полиномов, но не наоборот.